P-Vector Inverse Method

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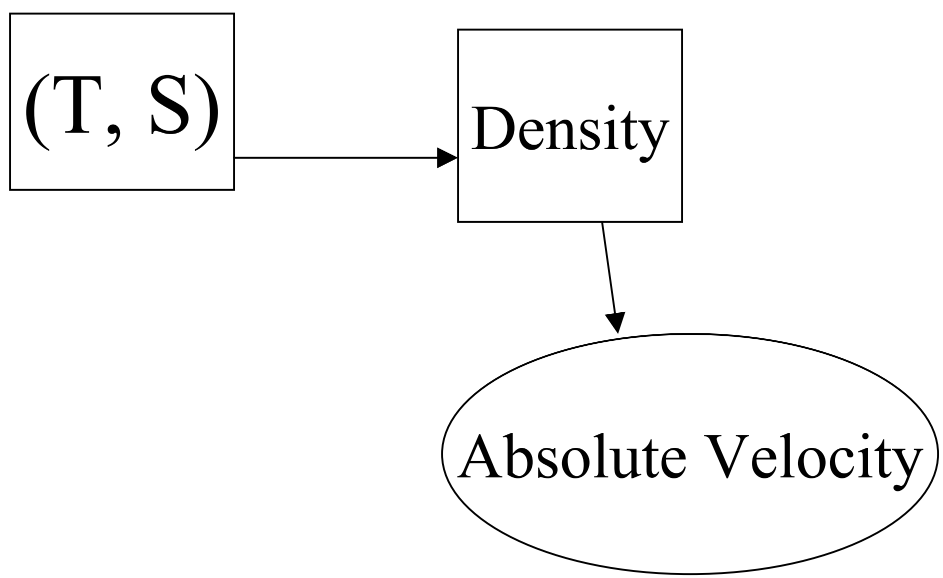
Purpose

- Obtaining absolute velocity from hydrographic data



Part-1 Inverse Method on z-Level

First Thought - Thermal Wind Relation

$$u = u_0 + \frac{g}{f\rho_0} \int_{z_0}^{z} \frac{\partial \rho}{\partial y} \, dz'$$

$$v = v_0 - \frac{g}{f\rho_0} \int_{z_0}^{z} \frac{\partial \rho}{\partial x} \, dz'$$

How to determine (u_0, v_0) ?

Level of No-Motion Assumption

- Assume that

$$(u_0, v_0) = 0$$

The level z_0 $\sim$ Reference level

How to determine z_0?

- (1) Arbitrarily chosen some lower level
  (downward decreasing current velocity)

- (2) Maximum common depth to which (T, S) measurements have been made

- (3) Core-level (e.g., oxygen minimum level,…)

Which level of no-motion?

Depth of the "zero level" (Nullfläche") in the Atlantic Ocean according to the assumption of different investigators

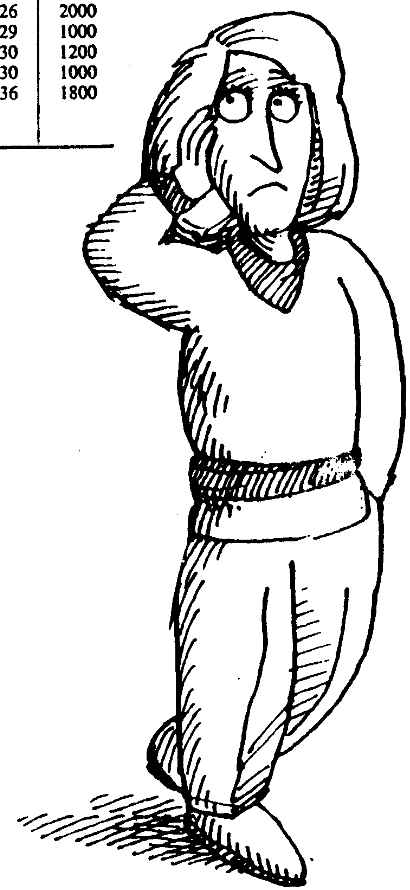
Investigator	Year	depth (m)	Investigator	Year	depth (m)
Bouquet de la Grye	1882	4000	Helland-Hansen and Nansen	1926	2000
Mohn	1885	550	Jacobsen	1929	1000
Zöppritz	1887	2000	Iselin	1930	1200
Wegemann	1899	1000	Helland-Hansen	1930	1000
Schott	1903	500	Iselin	1936	1800
Castens	1905	650			

500-m ?

1000-m?

1500-m?

2000-m?



Physical Basis for the P-Vector Inverse Method

- (1) Geostrophic Balance
- (2) Mass Conservation
- (3) No Major Cross-Isopycnal Mixing (except water masses in contact with the atmosphere)

Conservation of Mass and Potential Vorticity

$$\boldsymbol{V} \bullet \nabla \rho = 0$$

$$\boxed{\vec{V} \bullet \nabla q = 0, \qquad q \equiv f \frac{\partial \rho}{\partial z}}$$

Relationship Among Three Vectors

$\vec{V} \perp \nabla\rho$ $\vec{V} \perp \nabla q$

$$\vec{V} \sim \nabla q \times \nabla\rho$$

P-Vector

$$\vec{P} = \frac{\nabla q \times \nabla \rho}{|\nabla q \times \nabla \rho|}$$

$$\vec{P} \quad // \quad \vec{V}$$

$$\vec{V} = r(\lambda, \phi, z)\vec{P}$$

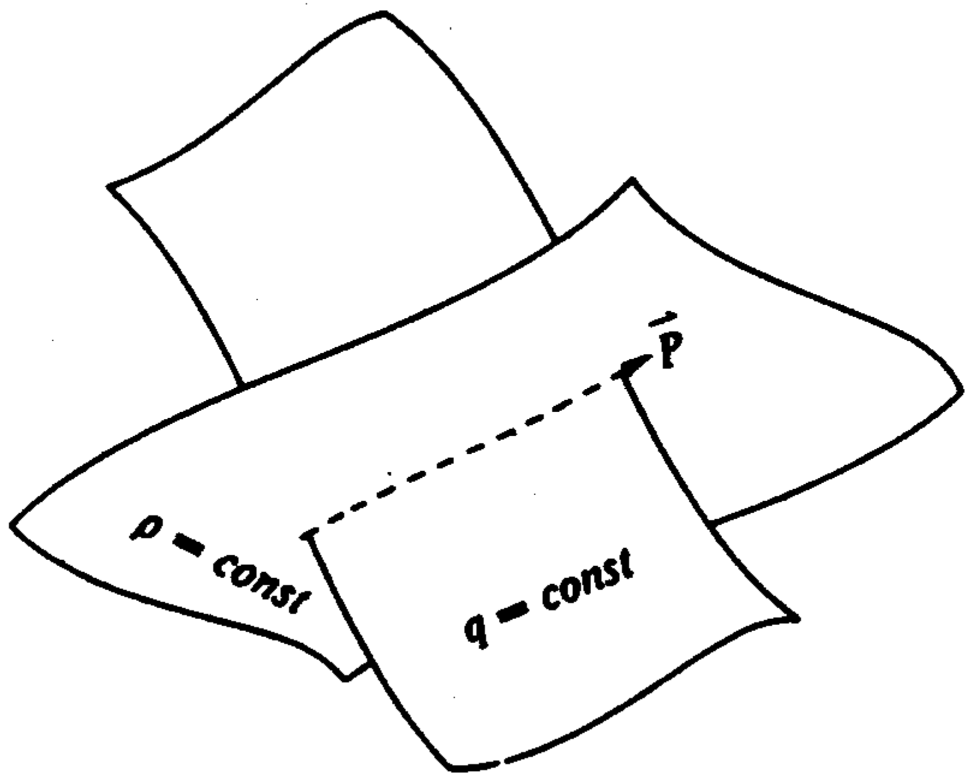
Two-Step Inverse Method

(1) Density determines the P-vector.

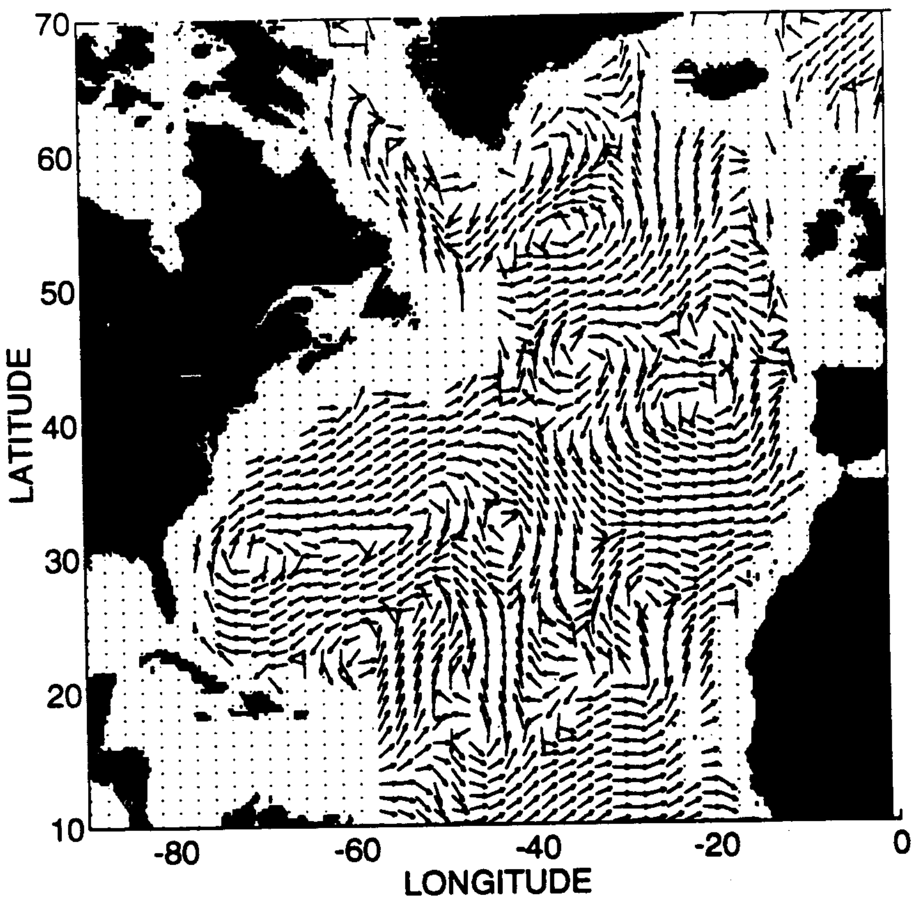
(2) Thermal wind relation determines γ.

P-Vector

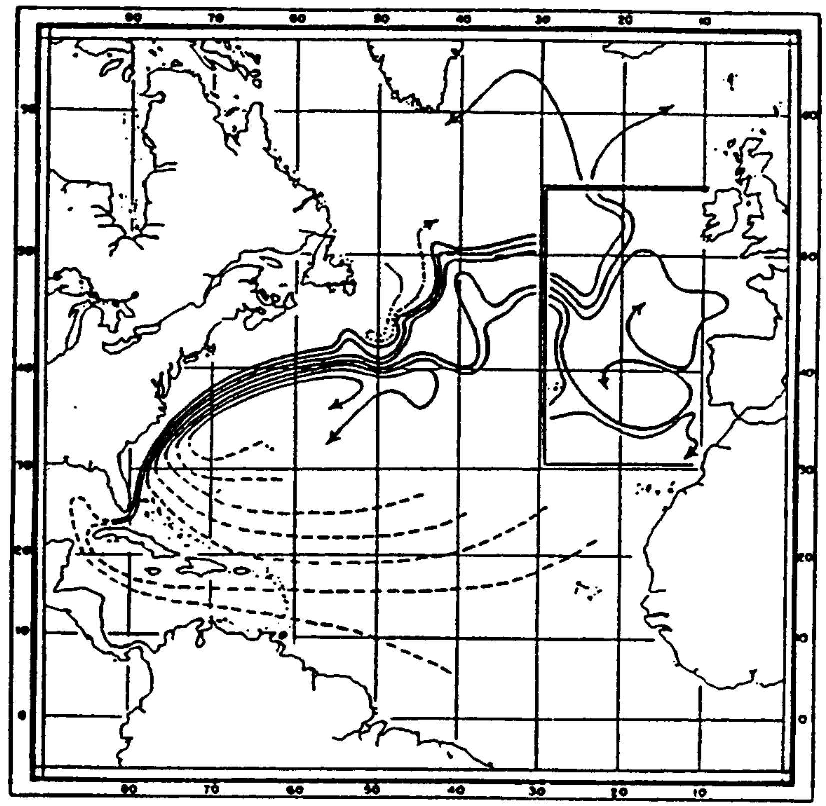
Intersection of density and potential vorticity surfaces



North Atlantic Horizontal P-Vector Field at 150 m Computed from NODC T, S Annual Mean Climatology (Chu 1995)



Classical View of Streamline Pattern (Iselin 1936)



Important Features Captured by the P-Vector Field

- (1) Anticyclonic subtropical gyre between 20^o-45^o N;
- (2) Recirculation cell on the western side (west of 40^oW) of the subtropical gyre;
- (3) Cyclonic-anticyclonic dipoles (30^o-10^oW, 30^o-50^oN);
- (4) High latitude cyclonic gyre (20^o-50^oW, 50^o-60^oN)

Thermal Wind Relation Determines γ

$$r^{(k)}P_x^{(k)} - r^{(m)}P_x^{(m)} = \Delta u_{km}$$

$$r^{(k)}P_y^{(k)} - r^{(m)}P_y^{(m)} = \Delta v_{km}$$

$$\Delta u_{km} \equiv \frac{g}{f\rho_0}\int_{z_m}^{z_k} \frac{\partial \rho}{\partial y} \, dz'$$

$$\Delta v_{km} \equiv -\frac{g}{f\rho_0}\int_{z_m}^{z_k} \frac{\partial \rho}{\partial x} \, dz'$$

Solution of γ

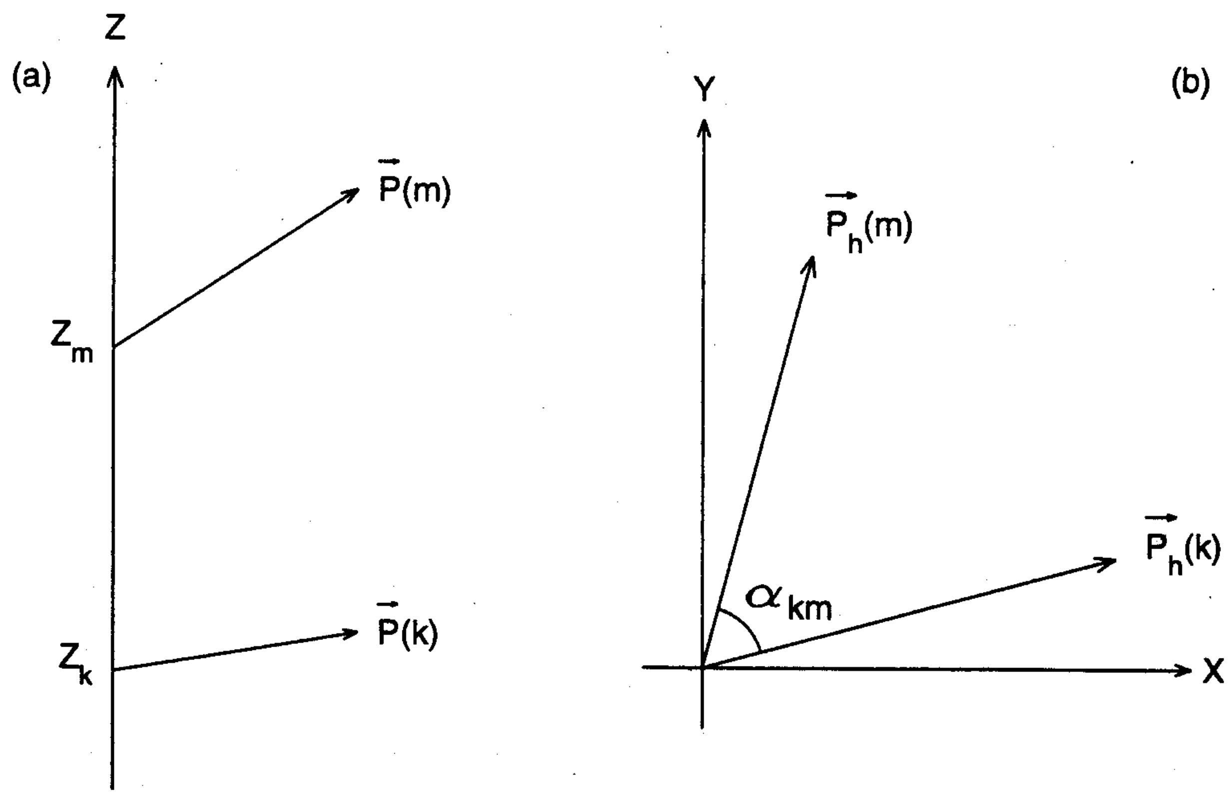
$$r^{(k)} = \frac{\begin{vmatrix} \Delta u_{km} & P_x^{(m)} \\ \Delta v_{km} & P_y^{(m)} \end{vmatrix}}{\sin(\alpha_{km})}$$

$$\begin{vmatrix} P_x^{(k)} & P_x^{(m)} \\ P_y^{(k)} & P_y^{(m)} \end{vmatrix} = \sin(\alpha_{km})$$

$$\alpha_{km} \neq 0$$

P-Vector Spiral

$\alpha_{km} \neq 0$



Theoretical Base of the P-Vector Method
Needler's Formula (1967)

$$\vec{V} = \frac{g[\vec{k} \bullet (\nabla q \times \nabla \rho)](\nabla \rho \times \nabla q)}{\nabla(f\partial q/\partial z) \bullet (\nabla q \times \nabla \rho)}$$

Necessary Conditions for the Validity of any Inverse Method

- (1) The ρ surface is not parallel to the q surface

$$\nabla q \times \nabla \rho \neq 0$$

- (2) Existence of velocity spiral

$$\begin{vmatrix} u^{(k)} & v^{(k)} \\ u^{(m)} & v^{(m)} \end{vmatrix} \neq 0$$

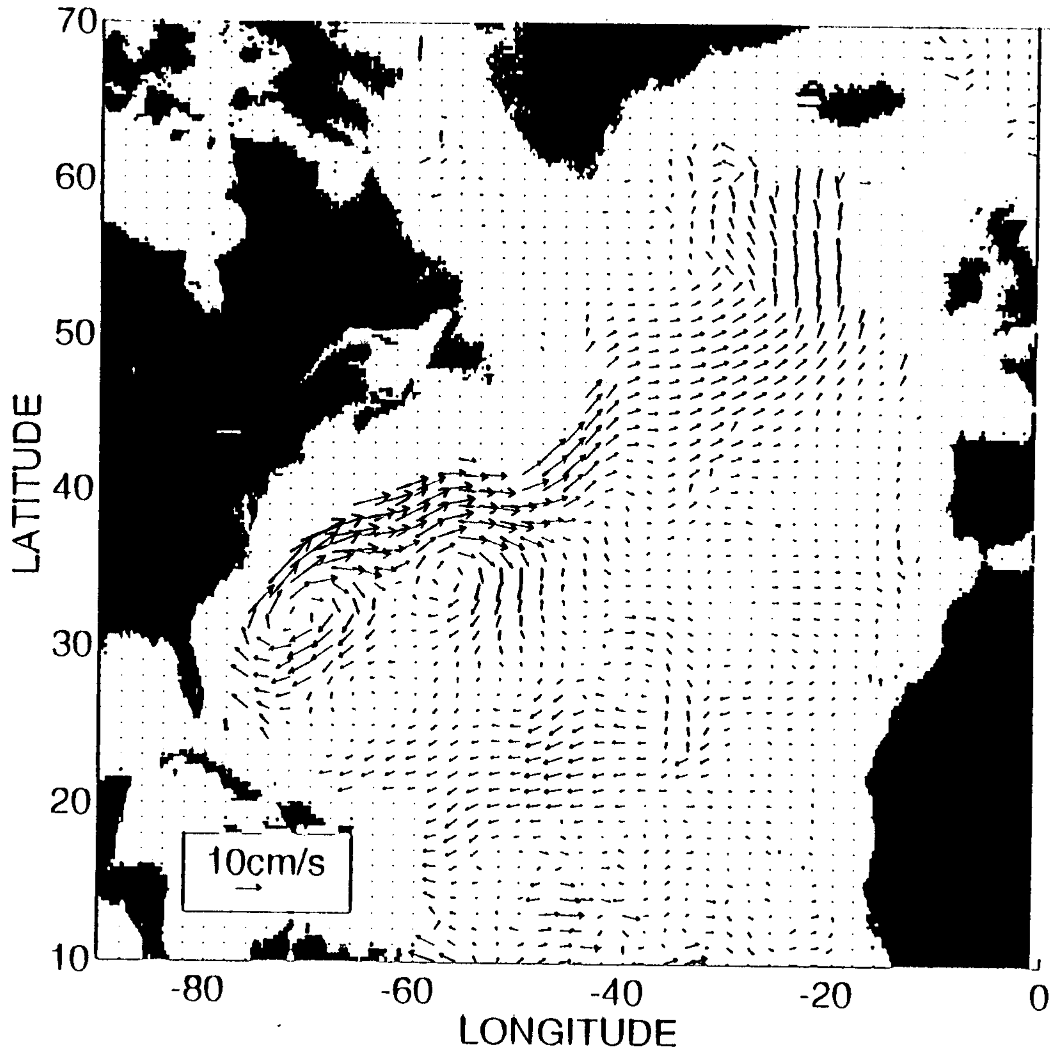
Necessary Conditions Using the P-vector

- (1) Existence of the P-vector

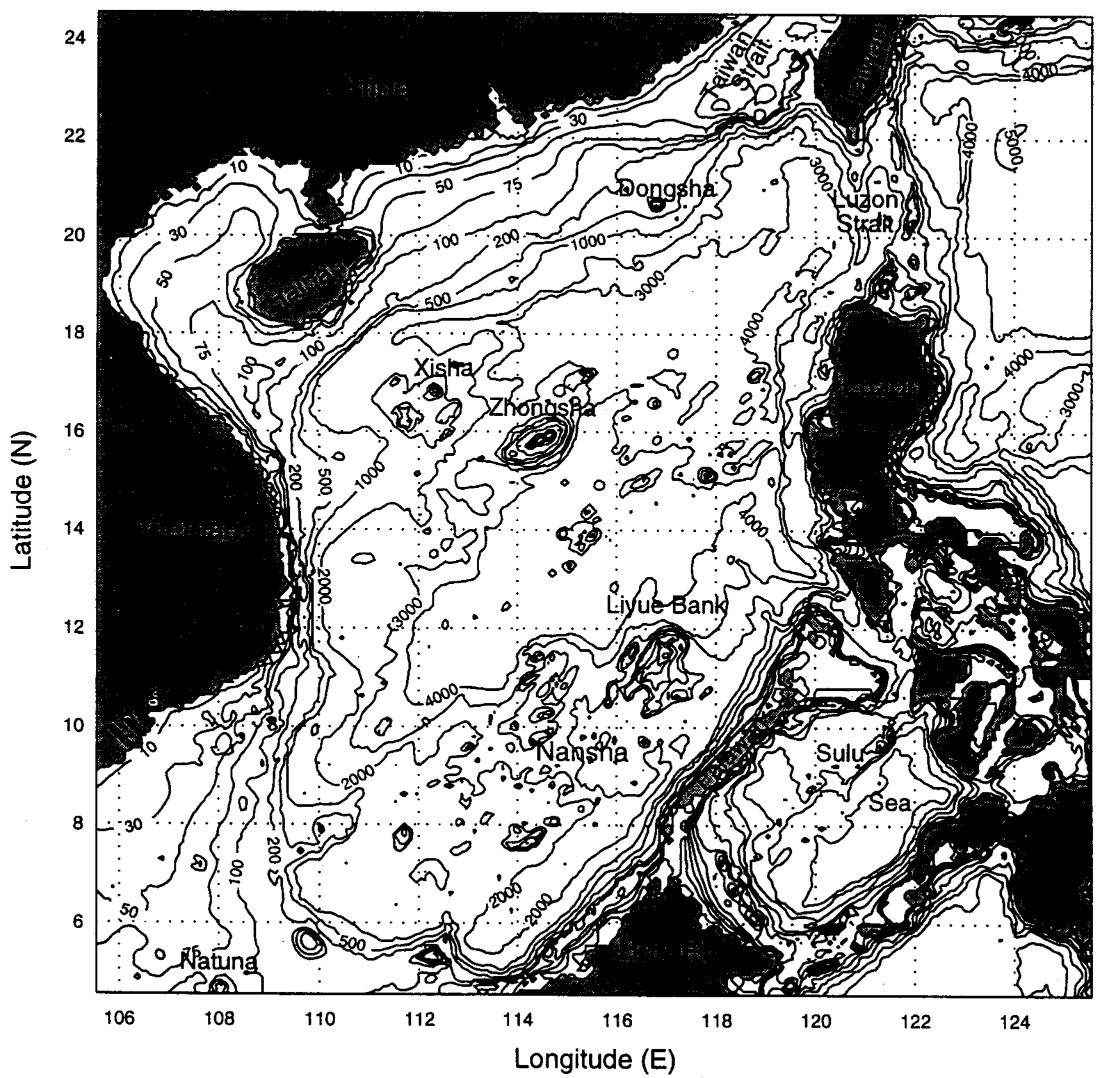
- (2) Existence of P-vector spiral

$$\alpha_{km} \neq 0$$

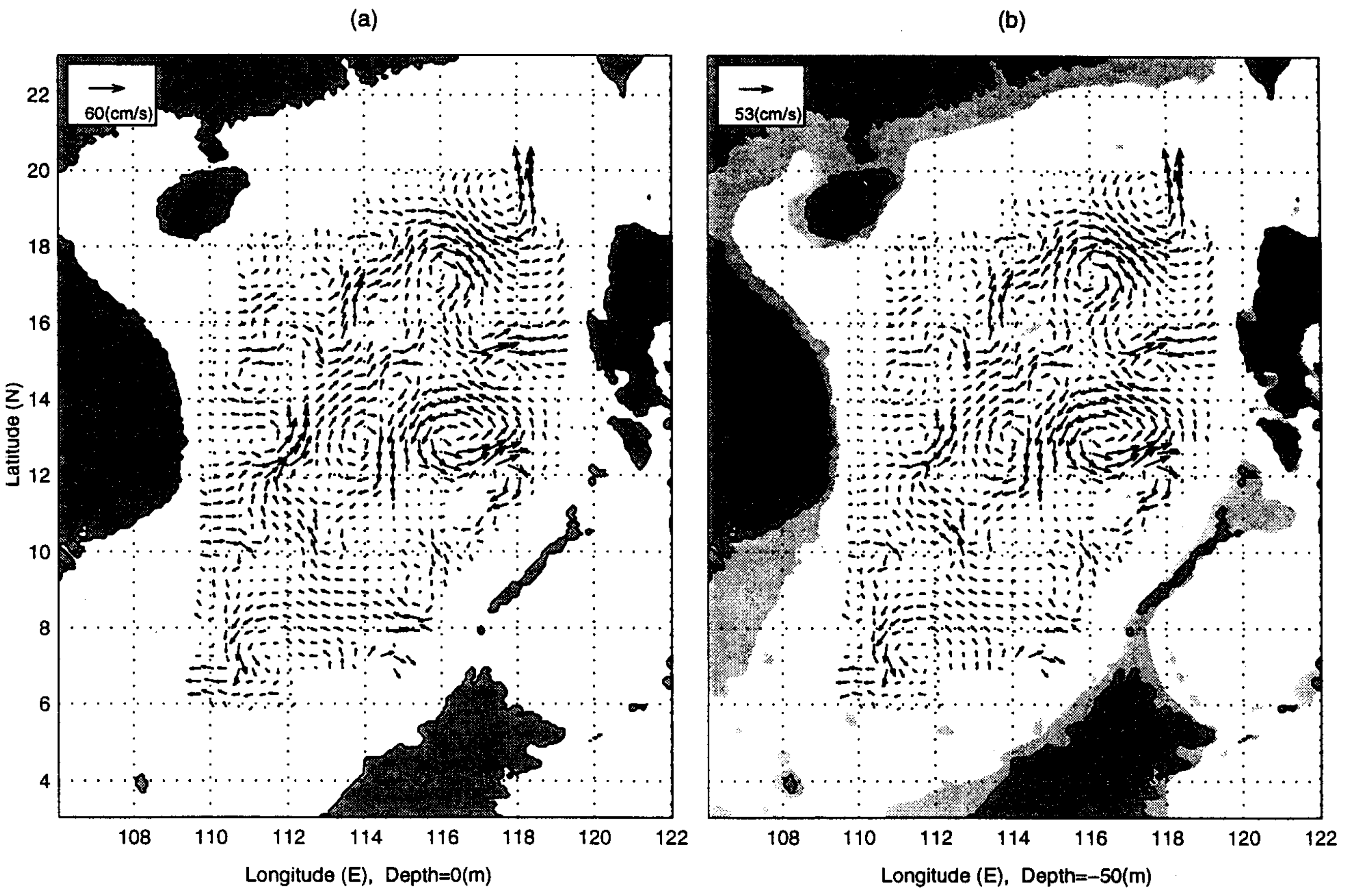
Example (1) – North Atlantic Circulation (500 m) Calculated from NODC data (Chu 1995)



Example –2 South China Sea Circulation May 1995 Calculated from AXBT and Monthly Mean Salinity



Example –2 South China Sea Circulation May 1995 Calculated from AXBT and Monthly Mean Salinity



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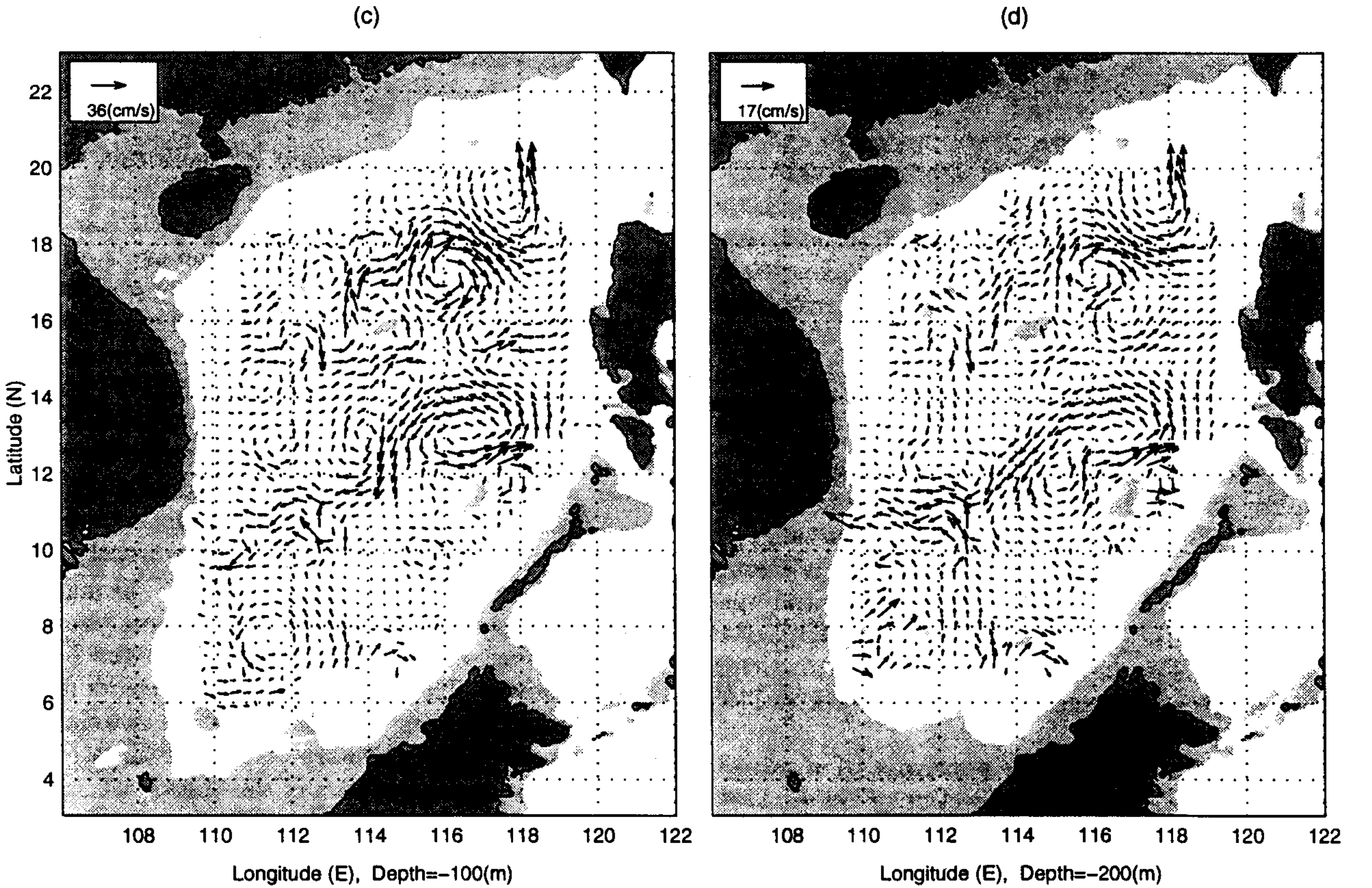
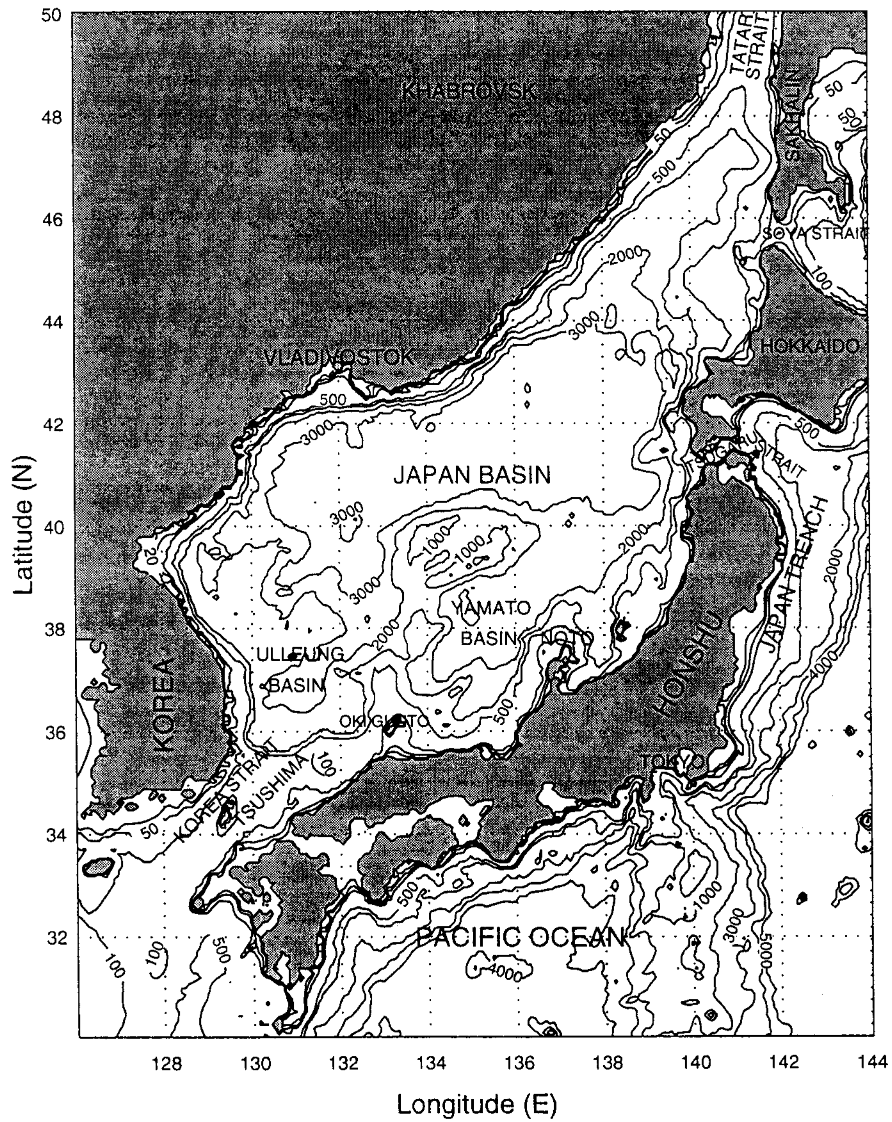


Example-3 Japan/East Sea (JES) Monthly Mean Circulations

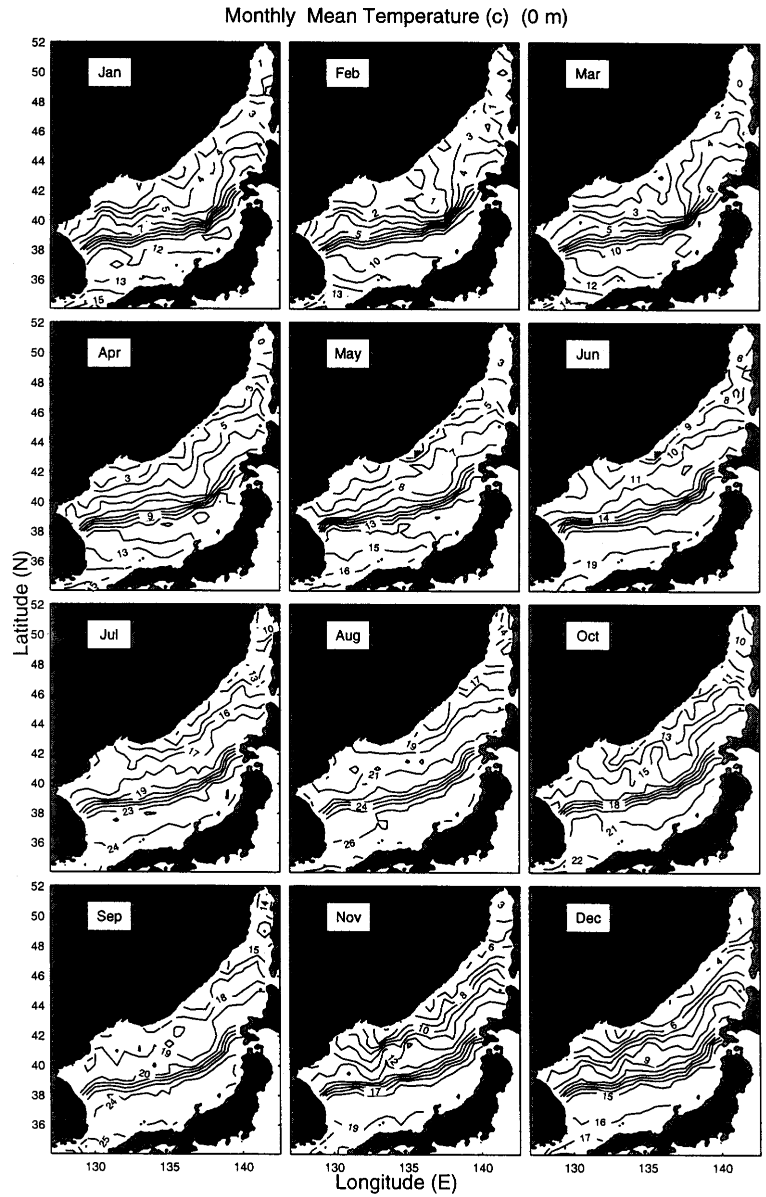


Example-3 Japan/East Sea (JES) Monthly Mean Circulations

Monthly Mean Temperature

at the Surface (0 m) From

GDEM

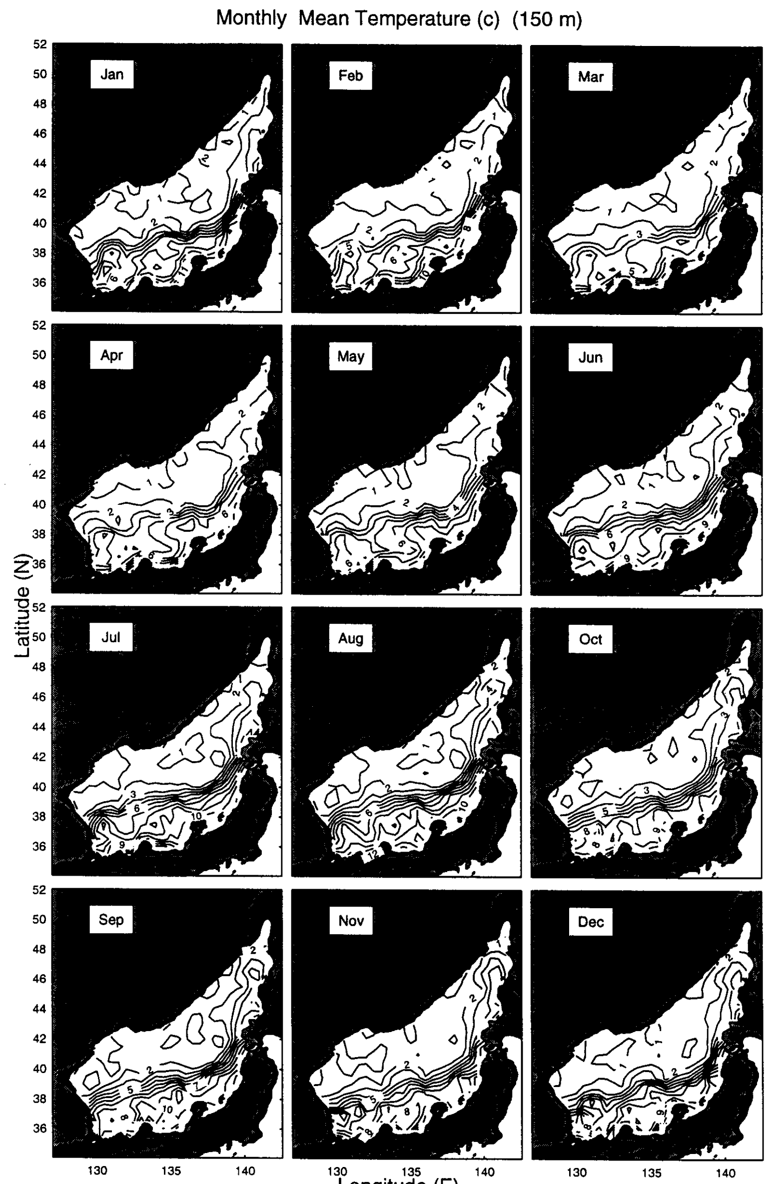


Example-3 Japan/East Sea (JES) Monthly Mean Circulations

Monthly Mean Temperature

at the Surface (150 m) From

GDEM

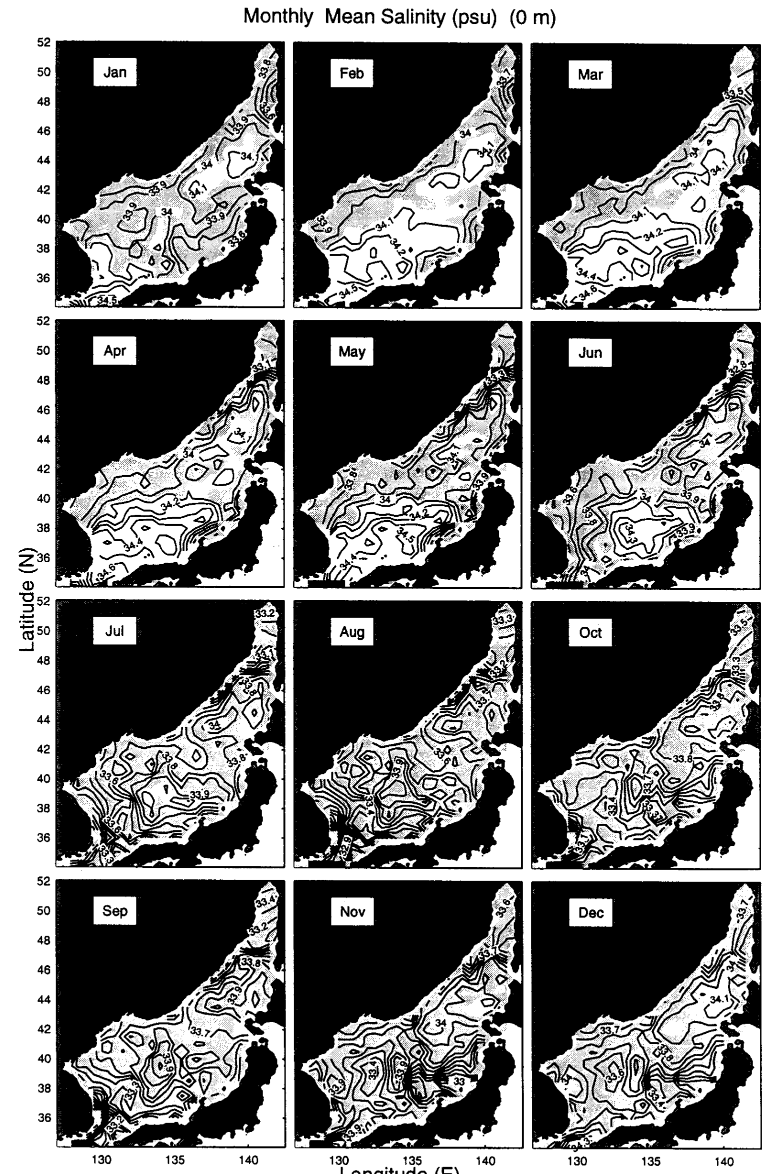


Example-3 Japan/East Sea (JES) Monthly Mean Circulations

Monthly Mean Salinity

at the Surface (0 m) From

GDEM

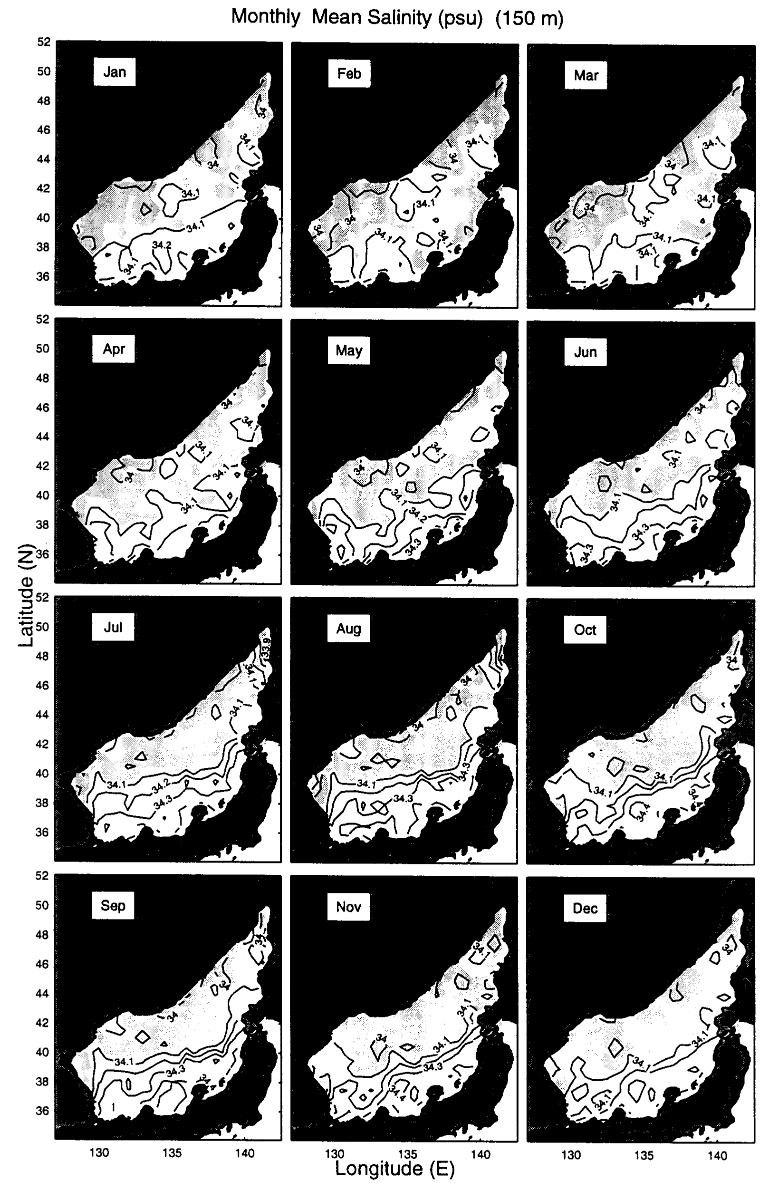


Example-3 Japan/East Sea (JES) Monthly Mean Circulations

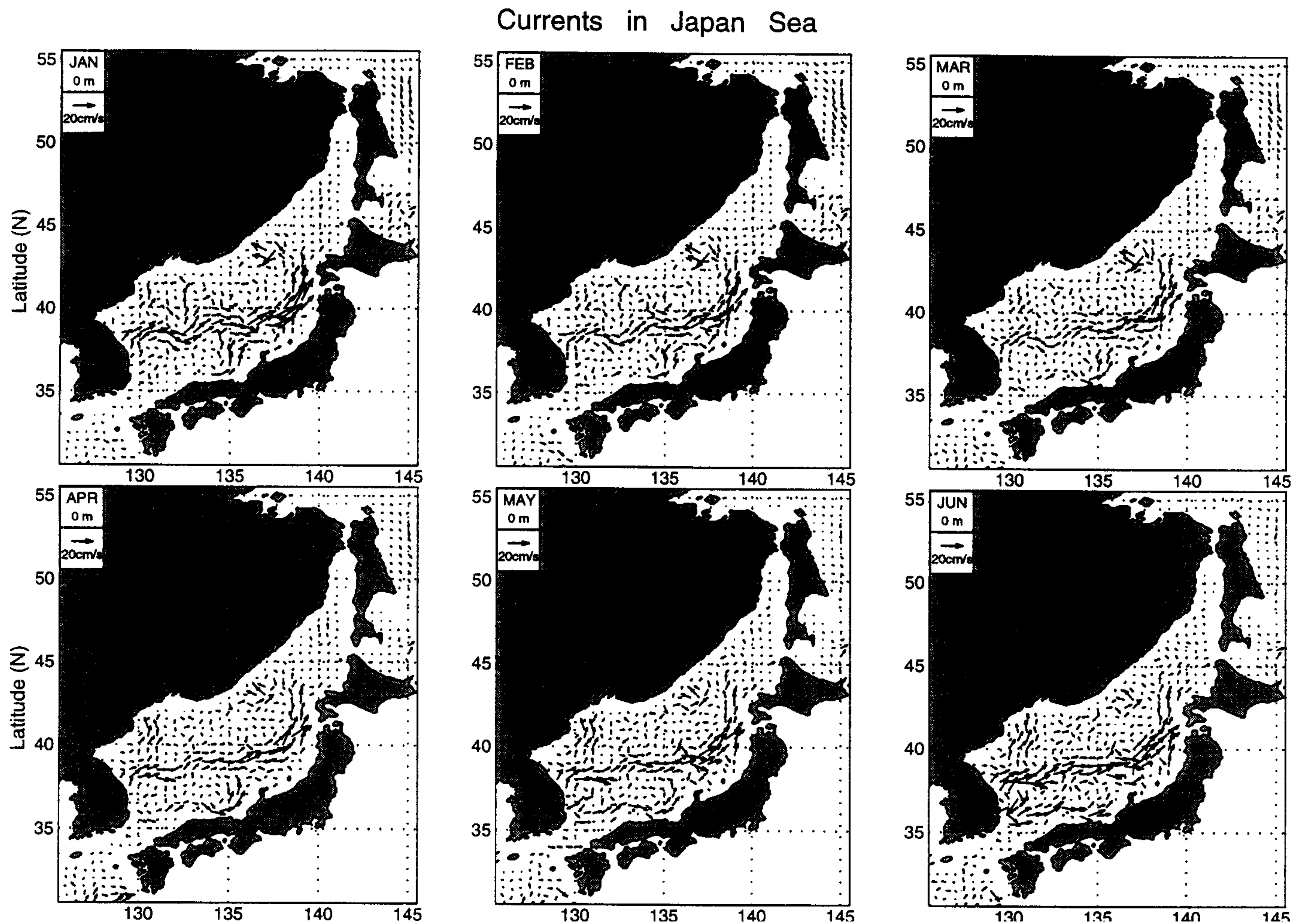
Monthly Mean Salinity

at the Surface (150 m) from

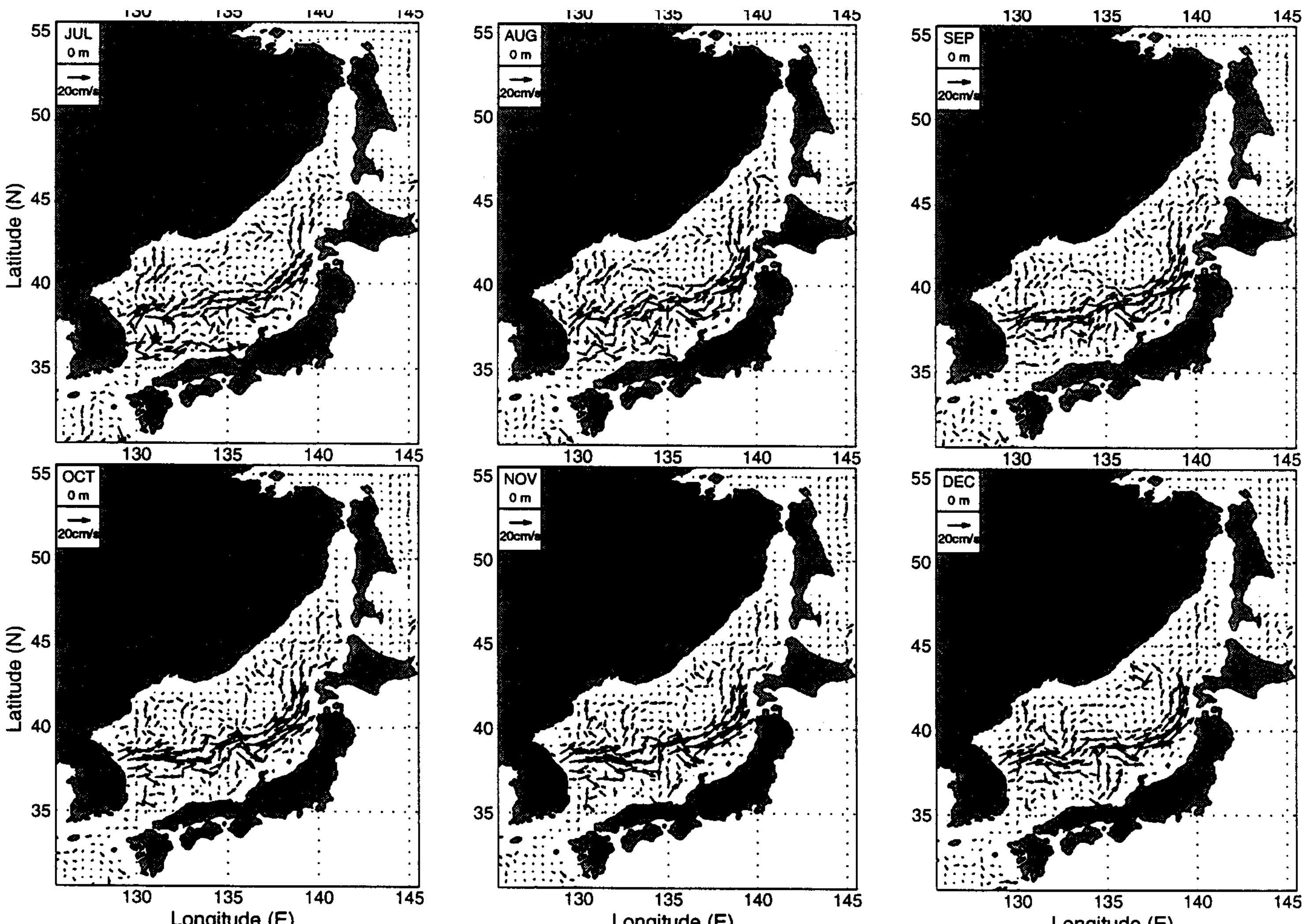
GDEM



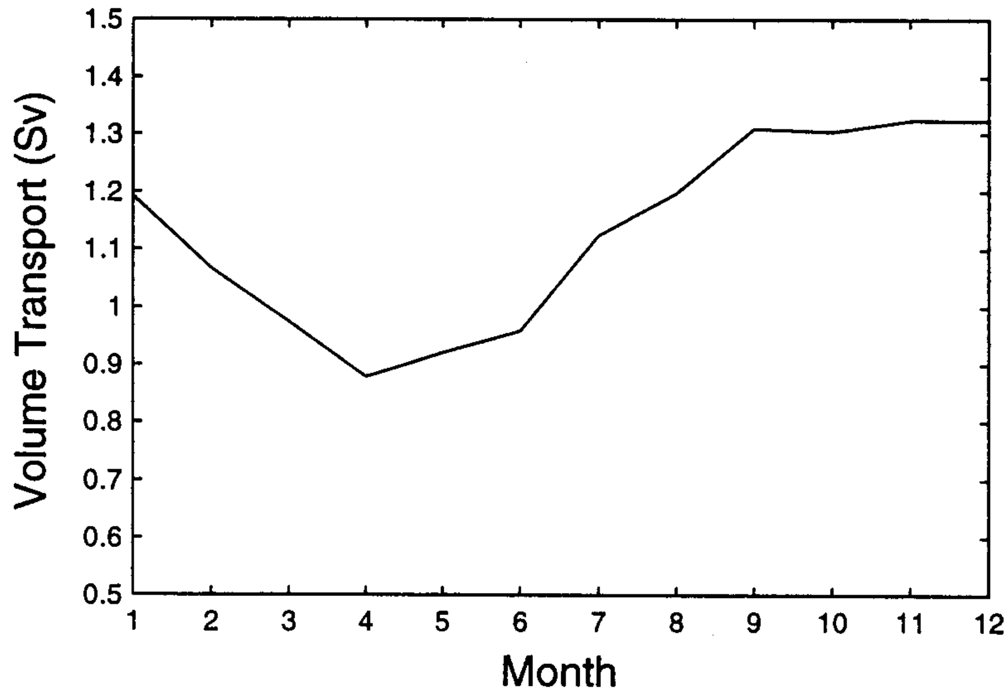
Example-3 Japan/East Sea (JES) Monthly Mean Circulations (0 m)



Example-3 Japan/East Sea (JES) Monthly Mean Circulations (0 m)



Volume Transport Through Korean/Tshushima Strait



Conclusions (Pat-1)

- (1) P-vector is a two-step inverse method with determination the orientation of the velocity first and the magnitude second.
- (2) P-vector method is easy to use and efficient.
- (3) P-vector method may have some problem in water mass conservation due to local determination.

Part-2 Variational P-Vector Method

Weakness of the P-Vector Method

Vertically Integrated Velocity

$$(U^{(P)}, V^{(P)}) = \int_{-H}^{0} (u^{(P)}, v^{(P)}) dz.$$

$(u^{(P)}, v^{(P)})$ are determined using the ordinary P-vector method.

Does not guarantee mass conservation over a domain (σ) due to local determination.

$$\int \int_{\sigma} \left[\frac{\partial U^{(P)}}{\partial x} + \frac{\partial V^{(P)}}{\partial y} \right] dx dy \neq 0$$

Variational Algorithm

Minimization of

$$J(U,V) = \frac{1}{2}\int\int_{\sigma}[(U - U^{(P)})^2 + (V - V^{(P)})^2]dxdy$$

with condition
$$\frac{\partial U^{(P)}}{\partial x} + \frac{\partial V^{(P)}}{\partial y} = 0.$$

Cost Function

$$L(U,V,\lambda) = J(U,V) + \int\int_{\sigma}\lambda\left[\frac{\partial U^{(P)}}{\partial x} + \frac{\partial V^{(P)}}{\partial y}\right]dxdy$$

Discretized Cost Function

$$\widehat{L} = \frac{1}{2}\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y-1}\left[(U_{ij}-U_{ij}^{(P)})^2+(V_{ij}-V_{ij}^{(P)})^2\right]\Delta x\Delta y$$

$$+\frac{1}{2}\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y-1}\lambda_{ij}(U_{ij}+U_{i,j-1}-U_{i-1,j}-U_{i-1,j-1})\Delta y$$

$$+\frac{1}{2}\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y-1}\lambda_{ij}(V_{ij}+V_{i-1,j}-V_{i,j-1}-V_{i-1,j-1})\Delta x$$

(2.1)

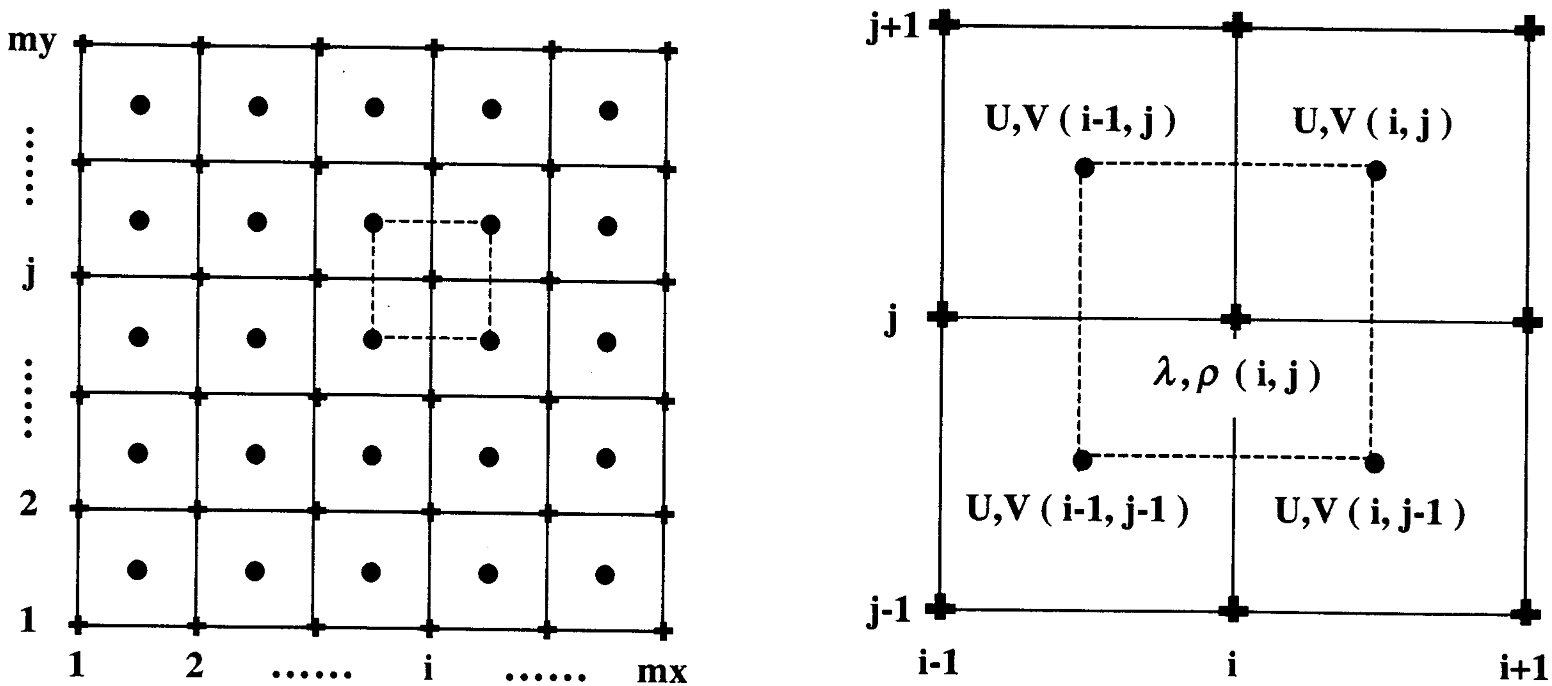
Global-Local Determination

Local: $U_{ij} \to U_{ij}^{(P)}, \qquad V_{ij} \to V_{ij}^{(P)}$

Global:

$$\frac{1}{\Delta x}(U_{ij} + U_{i,j-1} - U_{i-1,j} - U_{i-1,j-1}) + \frac{1}{\Delta y}(V_{ij} + V_{i-1,j} - V_{i,j-1} - V_{i-1,j-1}) \to 0$$

Grid Structure



● **U,V** points

✚ λ, ρ points

Optimal Determination of (U, V)

$$\frac{\partial \widehat{L}}{\partial U_{ij}} = 0, \quad \frac{\partial \widehat{L}}{\partial V_{ij}} = 0, \quad \frac{\partial \widehat{L}}{\partial \lambda_{ij}} = 0 \tag{2.2}$$

Substitution of (2.1) into (2.2) leads to

$$U_{ij} = U_{ij}^{(P)} - \frac{1}{2\Delta x}(\lambda_{ij} + \lambda_{i,j+1} - \lambda_{i+1,j} - \lambda_{i+1,j+1})$$

$$V_{ij} = V_{ij}^{(P)} - \frac{1}{2\Delta y}(\lambda_{ij} + \lambda_{i+1,j} - \lambda_{i,j+1} - \lambda_{i+1,j+1})$$

$$\frac{1}{\Delta x}(U_{ij} + U_{i,j-1} - U_{i-1,j} - U_{i-1,j-1}) + \frac{1}{\Delta y}(V_{ij} + V_{i-1,j} - V_{i,j-1} - V_{i-1,j-1}) = 0$$

(2.3)

Algebraic Equations for λ_{ij}

$$a_{11}\lambda_{i-1,j-1} + a_{21}\lambda_{i,j-1} + a_{31}\lambda_{i+1,j-1}$$

$$+a_{12}\lambda_{i-1,j} + a_{22}\lambda_{i,j} + a_{32}\lambda_{i+1,j}$$

$$+a_{13}\lambda_{i-1,j+1} + a_{23}\lambda_{i,j+1} + a_{33}\lambda_{i+1,j+1} = S_{ij} \tag{2.4}$$

ADI Method is used to solve (2.4)

Coefficients in (2.4)

$$a_{11} = a_{13} = a_{31} = a_{33} = -\frac{1}{4}\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right), \quad a_{22} = \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right),$$

$$a_{21} = a_{23} = -a_{12} = -a_{32} = \frac{1}{2}\left(\frac{1}{\Delta x^2} - \frac{1}{\Delta y^2}\right)$$

$$S_{ij} = \frac{1}{2\Delta x}\left(U_{ij}^{(P)} + U_{i,j-1}^{(P)} - U_{i-1,j}^{(P)} - U_{i-1,j-1}^{(P)}\right) + \frac{1}{2\Delta y}\left(V_{ij}^{(P)} + V_{i-1,j}^{(P)} - V_{i,j-1}^{(P)} - V_{i-1,j-1}^{(P)}\right).$$

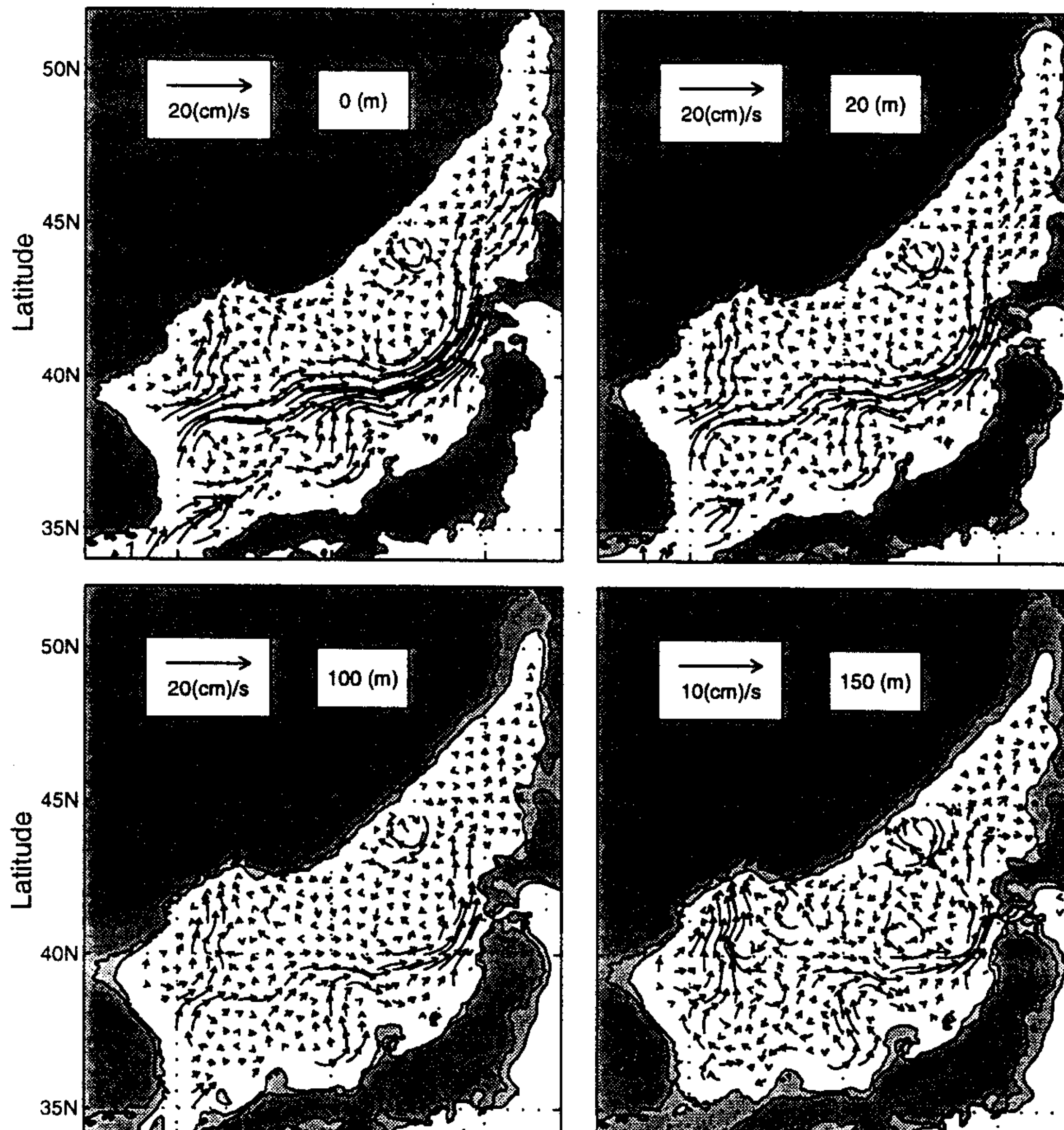
Updating Bottom Velocity

$$(u,v)_{-h} = \frac{1}{h}(U,V) - \frac{g}{fh\rho_0}\int_{-h}^{0} dz \int_{-h}^{z} \left(\frac{\partial\widehat{\rho}}{\partial y}, -\frac{\partial\widehat{\rho}}{\partial x}\right) dz'.$$

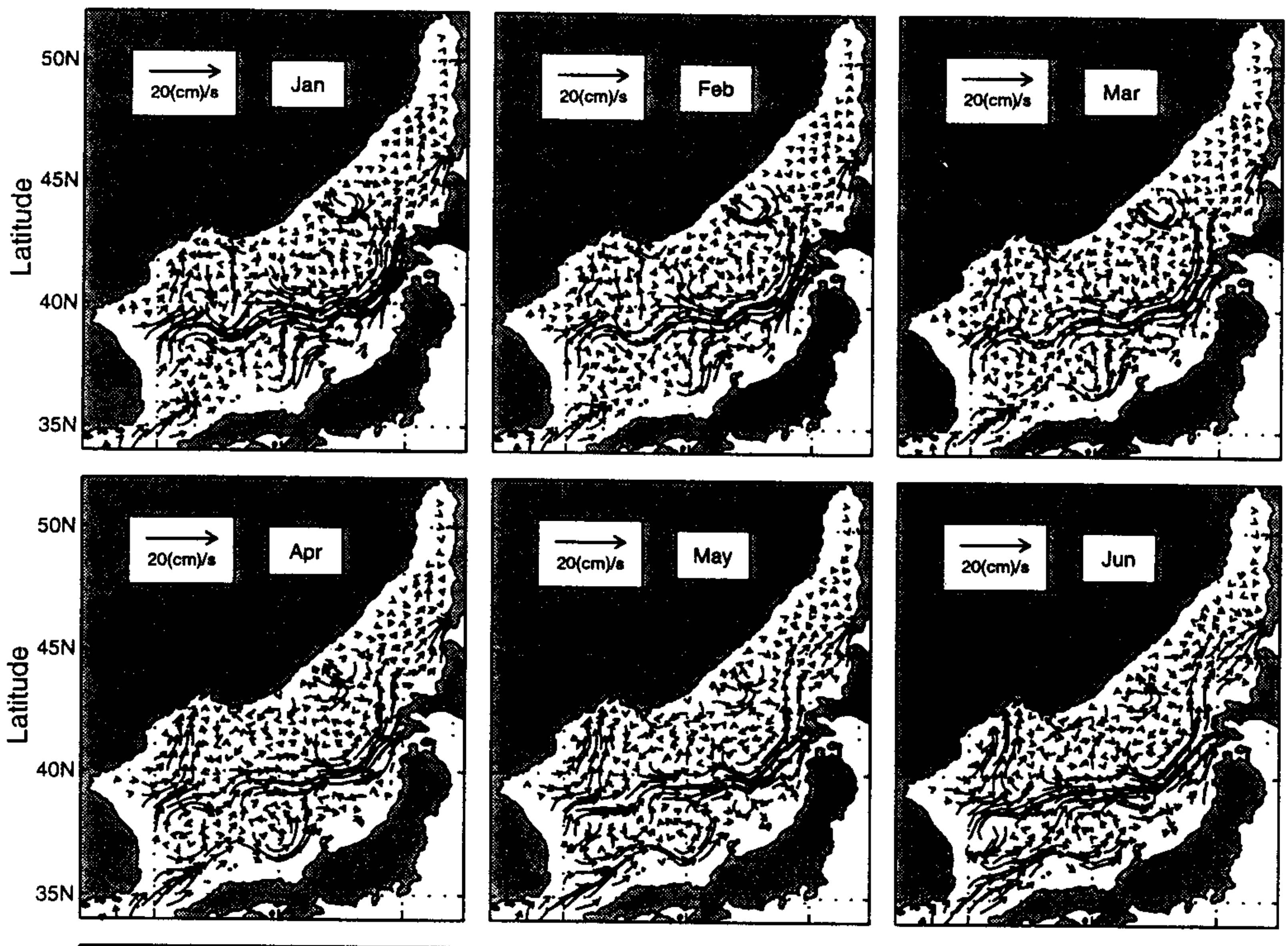
The absolute velocity is updated using the thermal wind relation with the bottom velocity.

Inverted Annual Mean JES Circulation

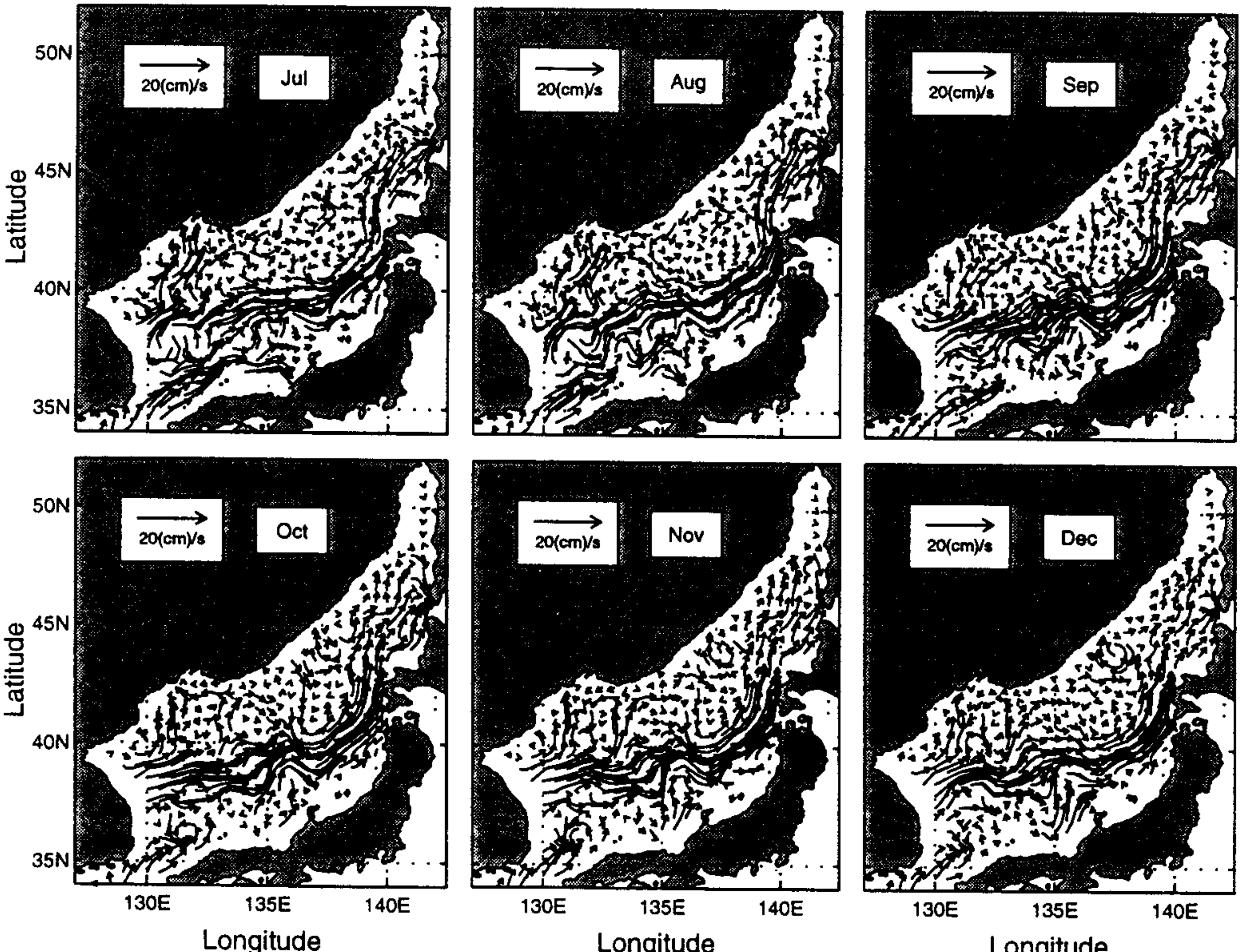
Note the improvement at the straits



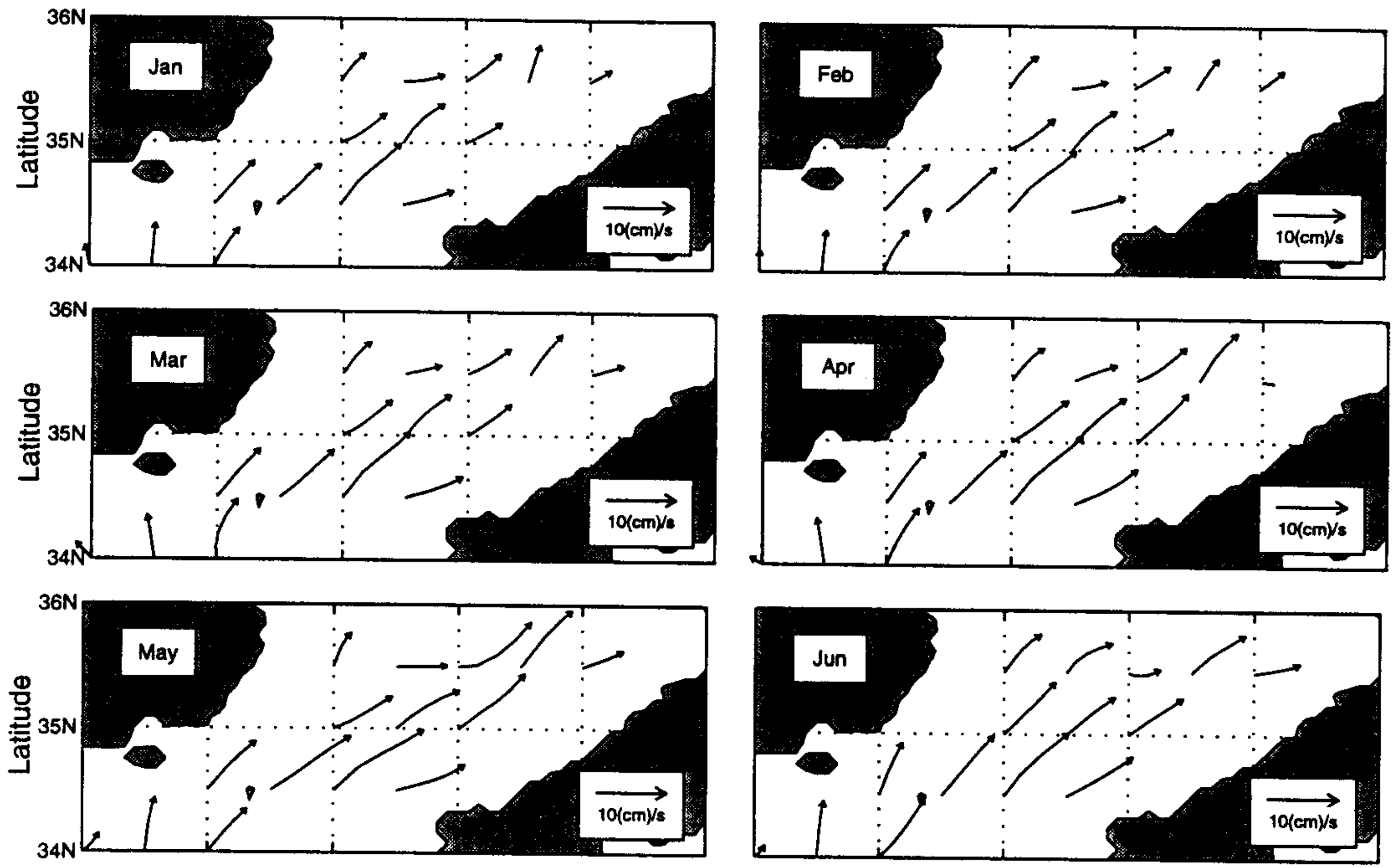
Monthly Mean Surface Circulation



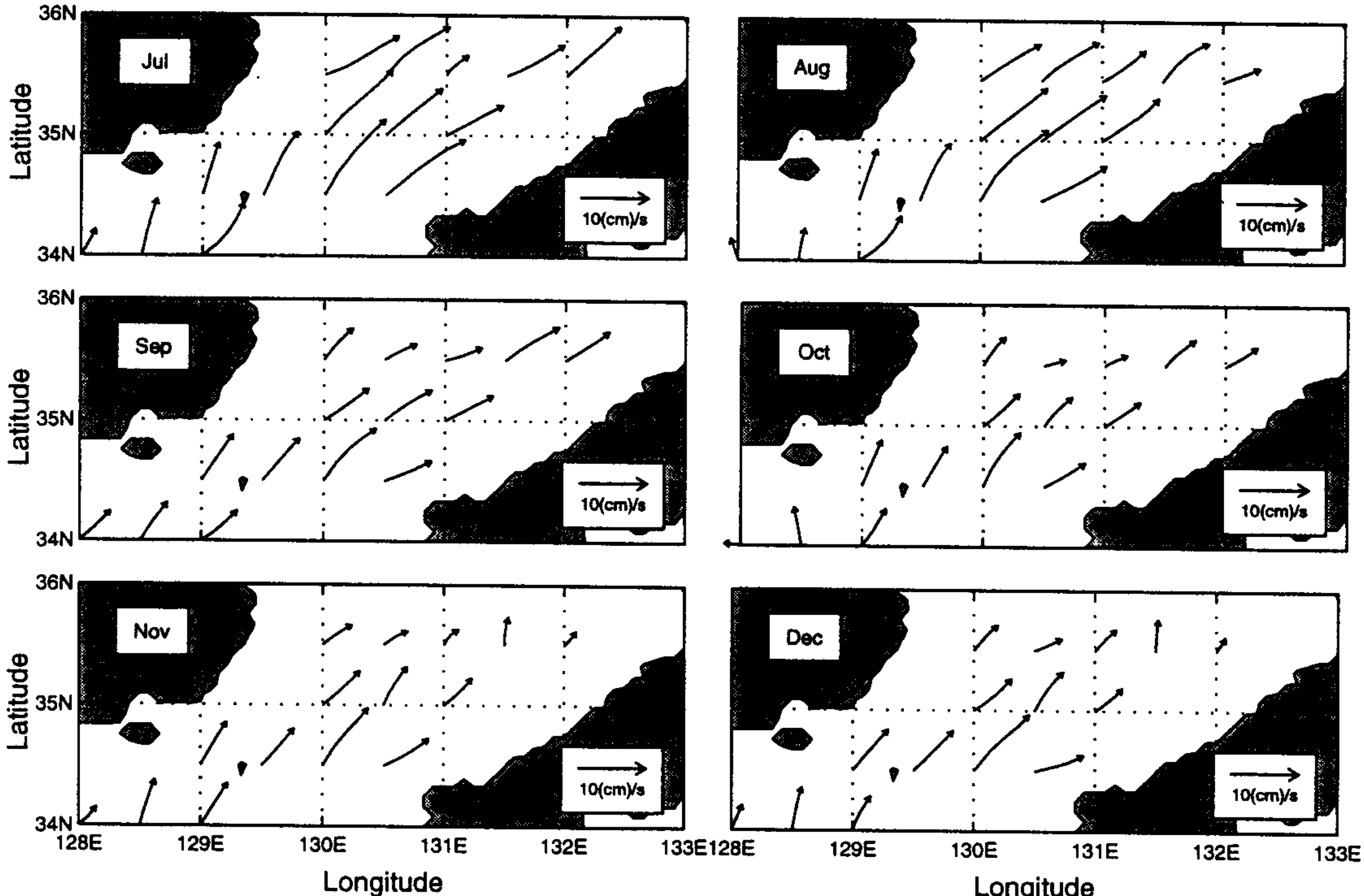
Monthly Mean Surface Circulation



Circulation Near Korean/Tsushima Strait



Circulation Near Korean/Tsushima Strait



Volume Transports Through Straits

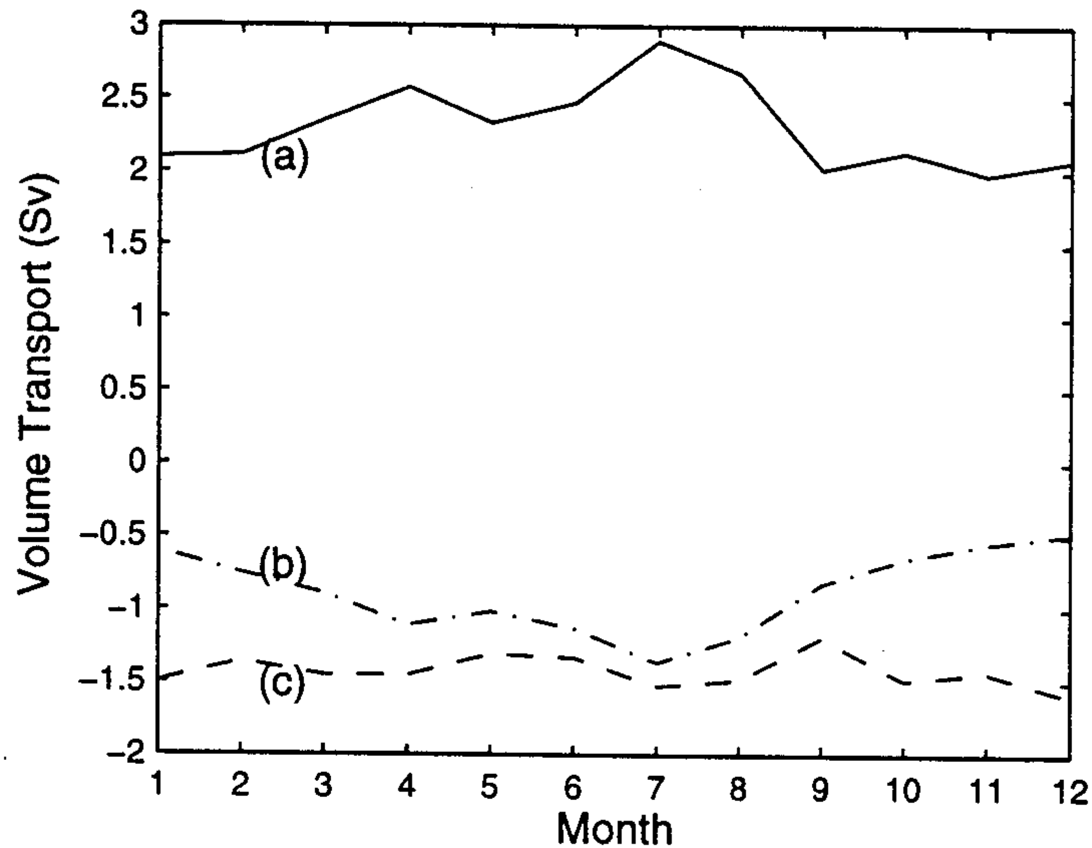
(a) Korean/Tsushima Strait

(b) Soya Strait

(c) Tsugaru Strait

Positive: into JES

Negative: out of JES



Volume Transports Through Straits (Sv)

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Soya	-0.6	-0.7	-0.9	-1.1	-1.0	-1.1	-1.4	-1.2	-0.8	-0.7	-0.5	-0.5
Tsugaru	-1.5	-1.4	-1.5	-1.5	-1.3	-1.3	-1.5	-1.5	-1.2	-1.5	-1.4	-1.6
Tsushima	2.1	2.1	2.4	2.6	2.3	2.4	2.9	2.7	2.0	2.2	1.9	2.1

Conclusions (Part-2)

- Variational P-vector method combines local-type (β-spiral, P-vector, …) and global-type (box model) inverse methods.

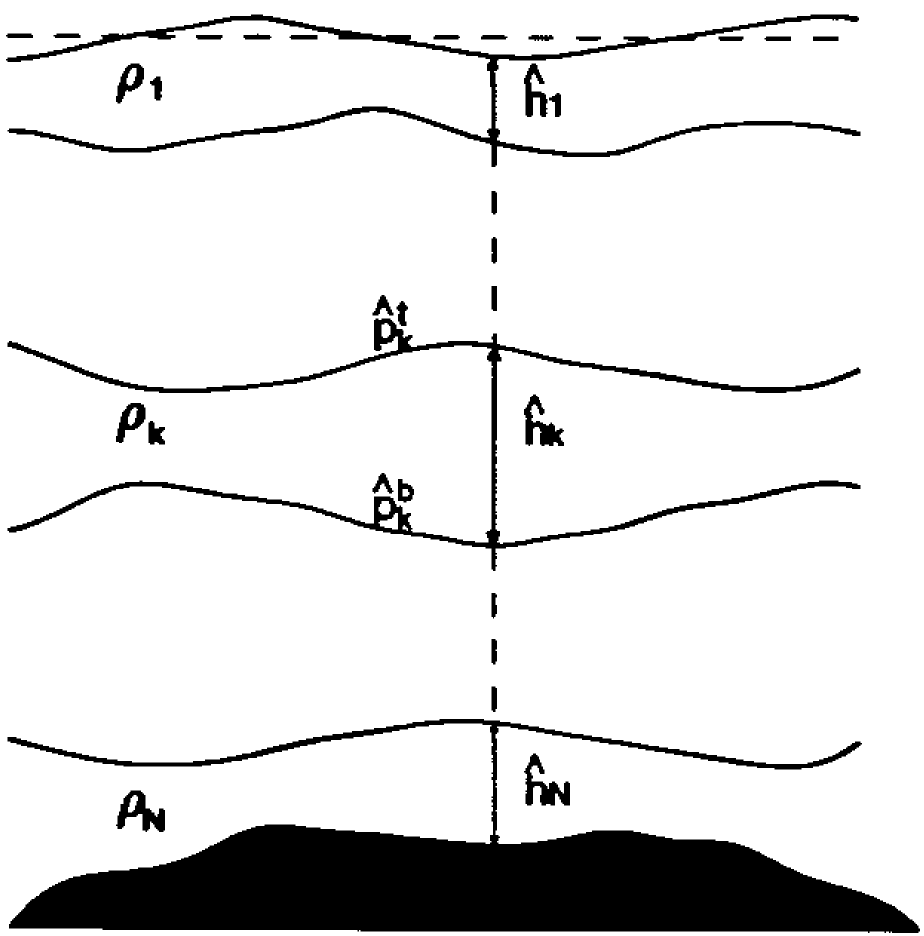
Part-3 P-Vector Inverse on Isopycnal Surface

Absolute Velocity

$$\boldsymbol{V} = \boldsymbol{V}_0 + \frac{1}{f}\boldsymbol{k} \times \int_{z_0}^{z} \nabla\left[\frac{\partial}{\partial z}\left(\frac{p}{\rho}\right)\right] dz',$$

$$\boldsymbol{V}' = \frac{1}{f}\boldsymbol{k} \times \int_{z_0}^{z} \nabla\left[\frac{\partial}{\partial z}\left(\frac{p}{\rho}\right)\right] dz' = -\frac{g}{f\rho_0}\boldsymbol{k} \times \int_{z_0}^{z} \nabla \rho \, dz'.$$

Isopycnal Surface



Isopycnal Surface

- Potential Density Surfaces (σ_θ) with the Depth $z^{(\sigma)}$

$$z^{(\sigma)} = R(x, y, \sigma).$$

- Vertical Distance Between two σ-Levels

$$h^{(\sigma)} = \frac{\partial z^{(\sigma)}}{\partial \sigma} \Delta\sigma.$$

Pseudo-Potential Vorticity Conservation on Isopycnal Surface (McDogall 1988)

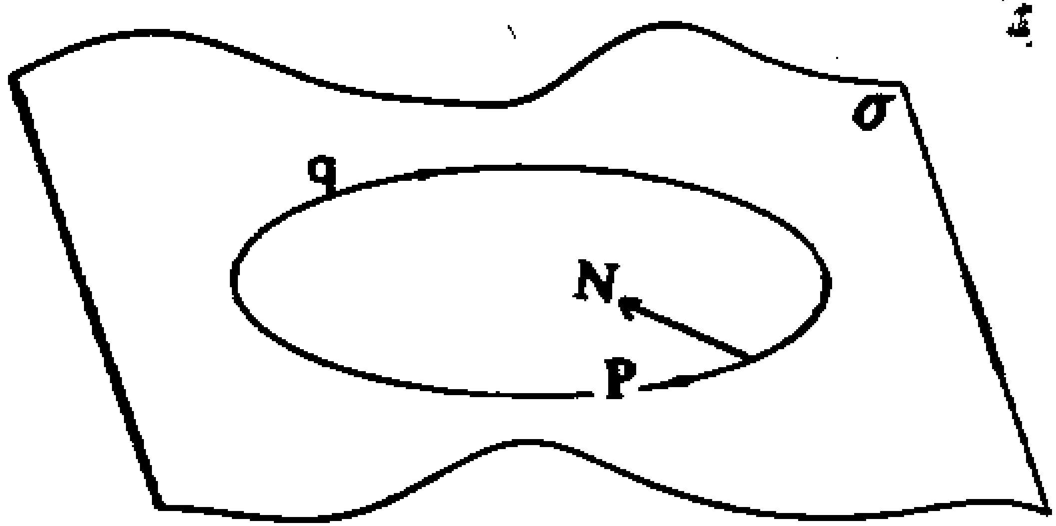
- Pseudo Potential Vorticity

$$q^{(\sigma)} = \ln\left[Q^{(\sigma)}\right], \quad Q^{(\sigma)} = \frac{f}{h^{(\sigma)}}$$

- Conservation of Seudo Potential Vorticity on Isopycnal Surafce

$$V^{(\sigma)} \bullet \nabla_{\sigma}\left[q^{(\sigma)}\right] = \frac{\partial w^{(\sigma)}}{\partial z},$$

Two Unit Vectors on Isopycnal Surface



Two Unit Vectors on Isopycnal Surface

- P – Vector

$$\boldsymbol{P} = \frac{1}{\left|\nabla q^{(\sigma)}\right|}\left(\frac{\partial q^{(\sigma)}}{\partial y}\boldsymbol{i} - \frac{\partial q^{(\sigma)}}{\partial x}\boldsymbol{j}\right),$$

- N - Vector

$$\boldsymbol{N} = \frac{\nabla_{\sigma}\left(q^{(\sigma)}\right)}{\left|\nabla_{\sigma}\left(q^{(\sigma)}\right)\right|}$$

P-Vector Components

$$P_x = \left(\frac{\beta}{f} - \frac{\partial \ln h^{(\sigma)}}{\partial y}\right) \Bigg/ \left[\left(\frac{\beta}{f} - \frac{\partial \ln h^{(\sigma)}}{\partial y}\right)^2 + \left(\frac{\partial \ln h^{(\sigma)}}{\partial x}\right)^2\right]^{1/2}.$$

$$P_y = \frac{\partial \ln h^{(\sigma)}}{\partial x} \Bigg/ \left[\left(\frac{\beta}{f} - \frac{\partial \ln h^{(\sigma)}}{\partial y}\right)^2 + \left(\frac{\partial \ln h^{(\sigma)}}{\partial x}\right)^2\right]^{1/2}$$

Absolute Velocity on Isopycnal Surface

- With Diapycnal Velocity

$$V^{(\sigma)} = \gamma P + \frac{\partial w^{(\sigma)} / \partial z}{\left|\nabla_{\sigma}\left(q^{(\sigma)}\right)\right|} N$$

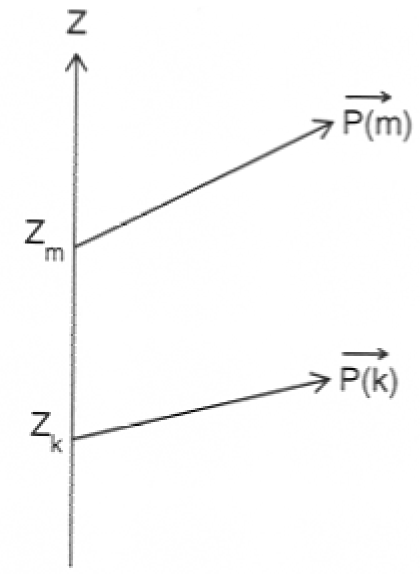
- Without Diapycnal Velocity

$$V^{(\sigma)} = \gamma P$$

P-Vector Inverse Method

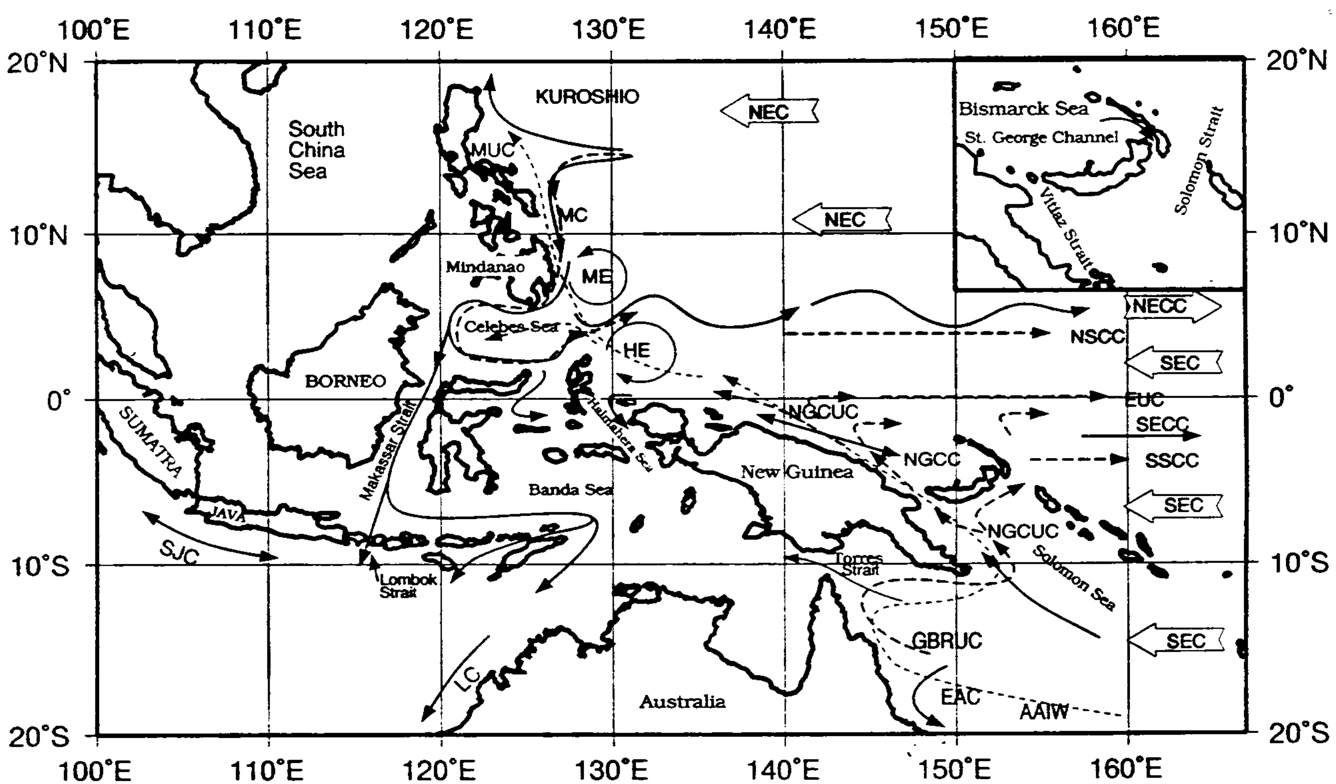
$$\gamma^{(k)} P_x^{(k)} - \gamma^{(m)} P_x^{(m)} = \Delta u_{km}$$

$$\gamma^{(k)} P_y^{(k)} - \gamma^{(m)} P_y^{(m)} = \Delta v_{km}$$



Example-1
Water Mass Crossroads

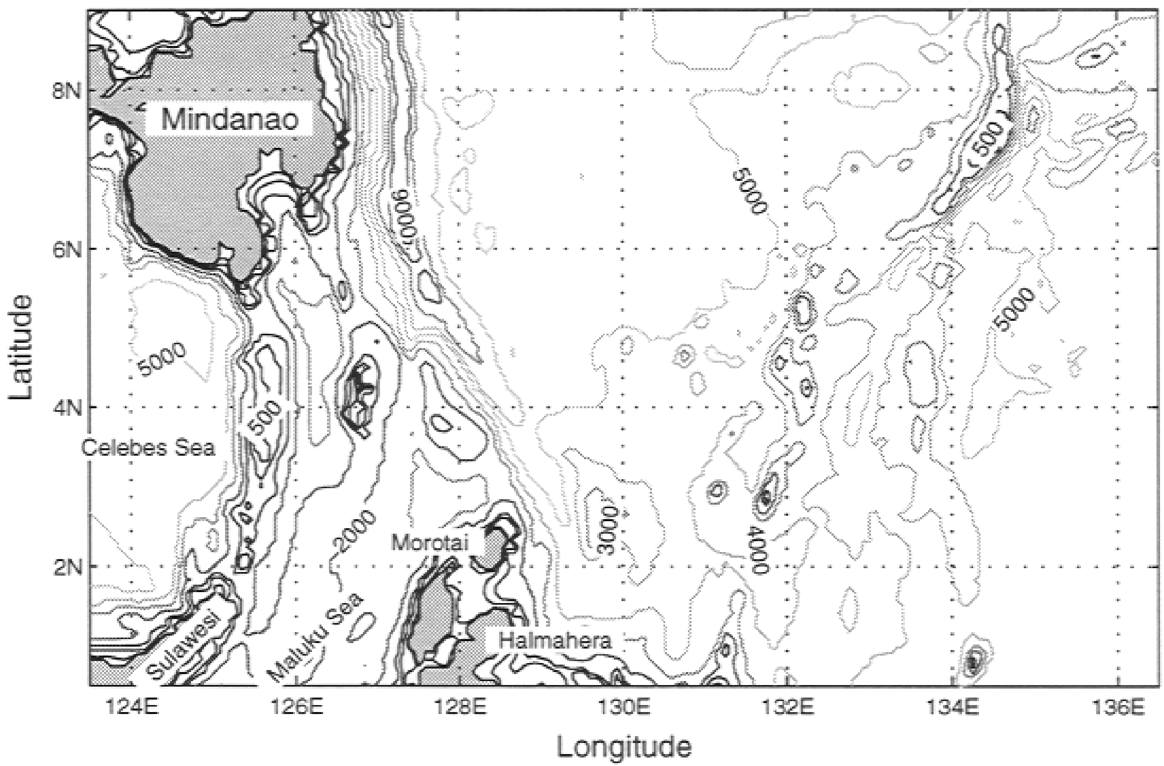
Water Mass Crossroads (Fines et al. 1994)



Literature

- Lukas (1988, JGR)
- Lukas et al. (1991, 1996)
- Godfrey (1996)
- Fine et al. (1994)
- Qiu and Lukas (1999)
- Qiu et al. (1998)
- Qu (1998. 1999)
- Wajsowicz (1993, 1999a, b)

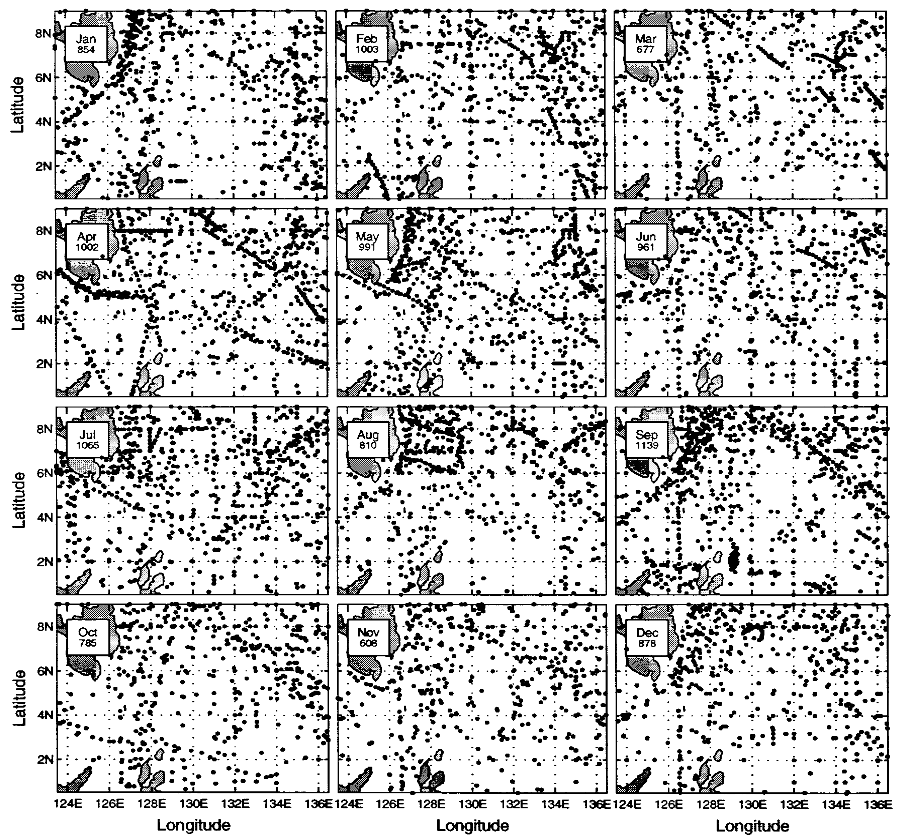
Geography and Topography



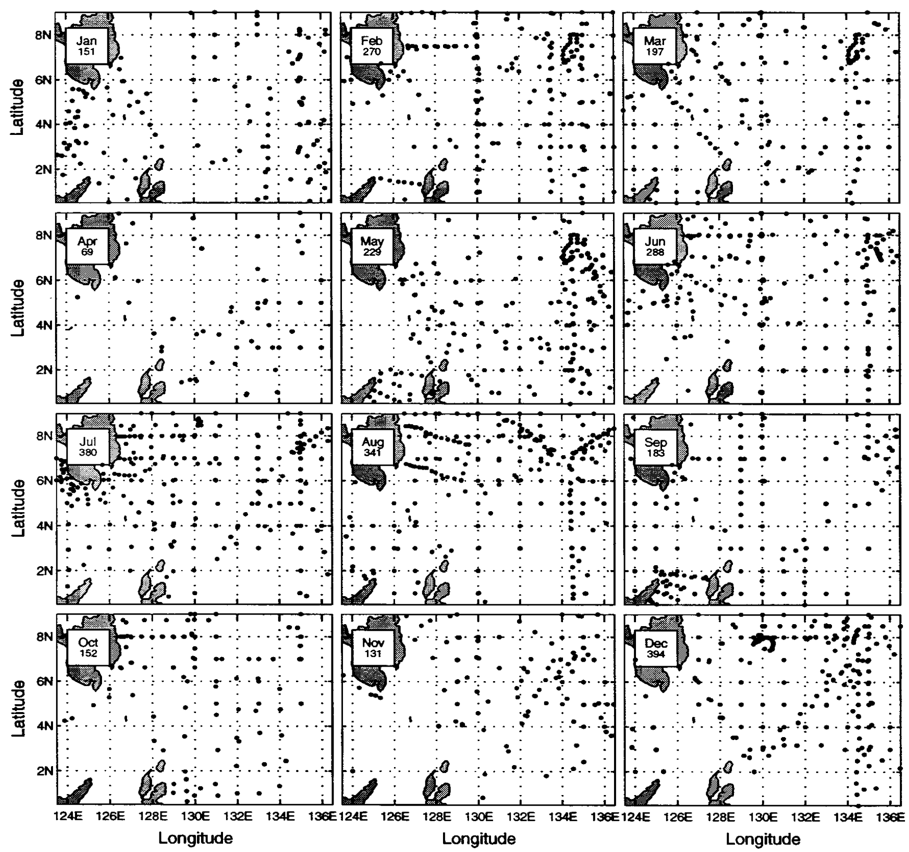
Navy’s GDEM Climatology

- Monthly mean temperature and salinity
- 0.5^o Horizontal resolution
- 59 Vertical levels
- Built-up from The Navy’s Master Oceanographic Observational Data Set (MOODS)

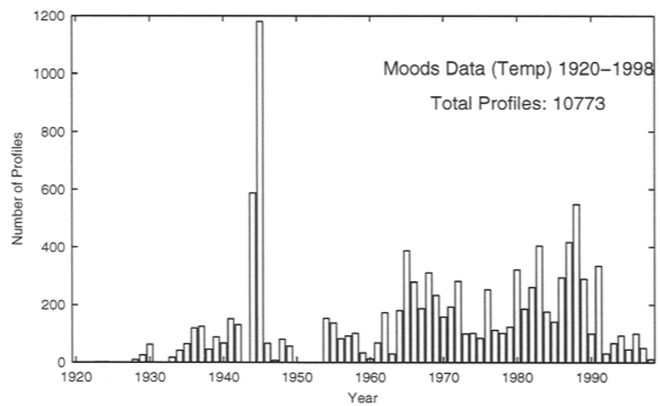
Navy's MODDS (Temperature)

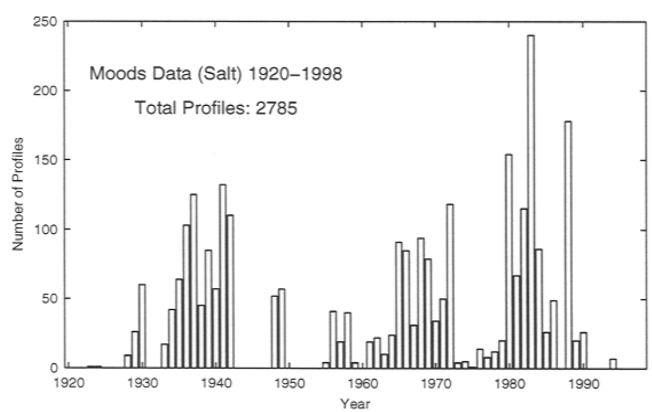


MOODS Data (Salinity)

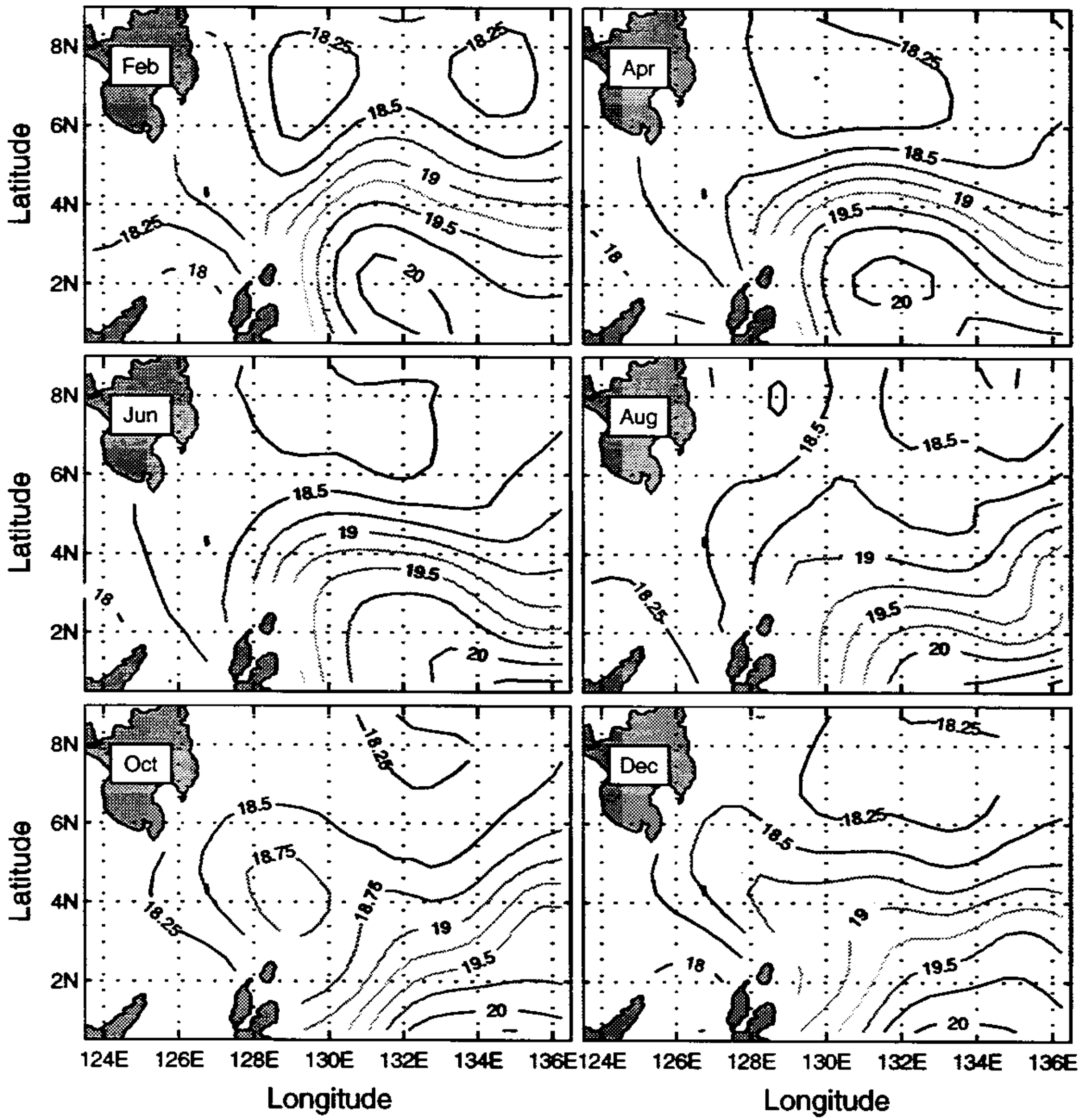


Temporal Distributions of MOODS Stations

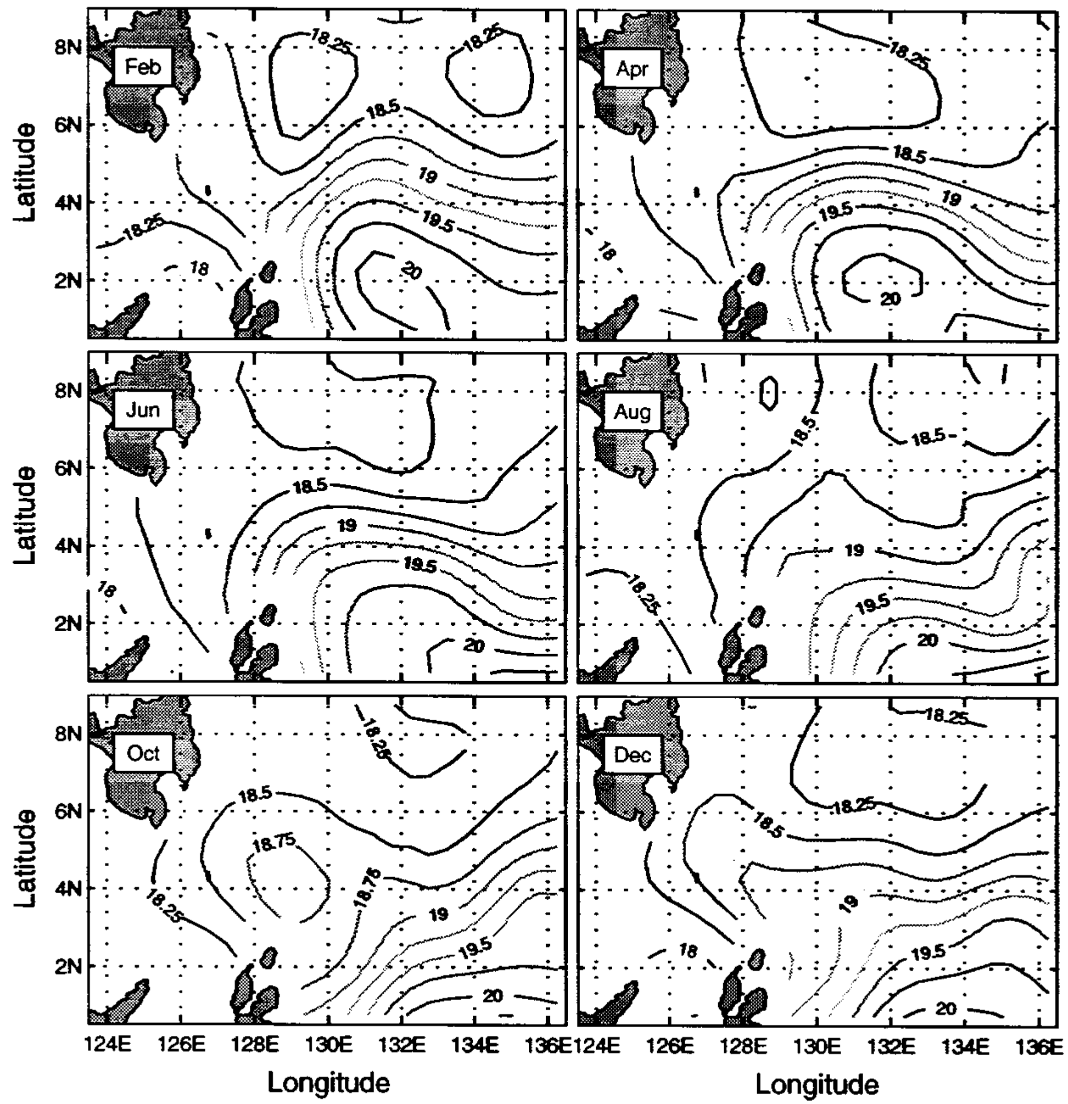




Temperature (GDEM)

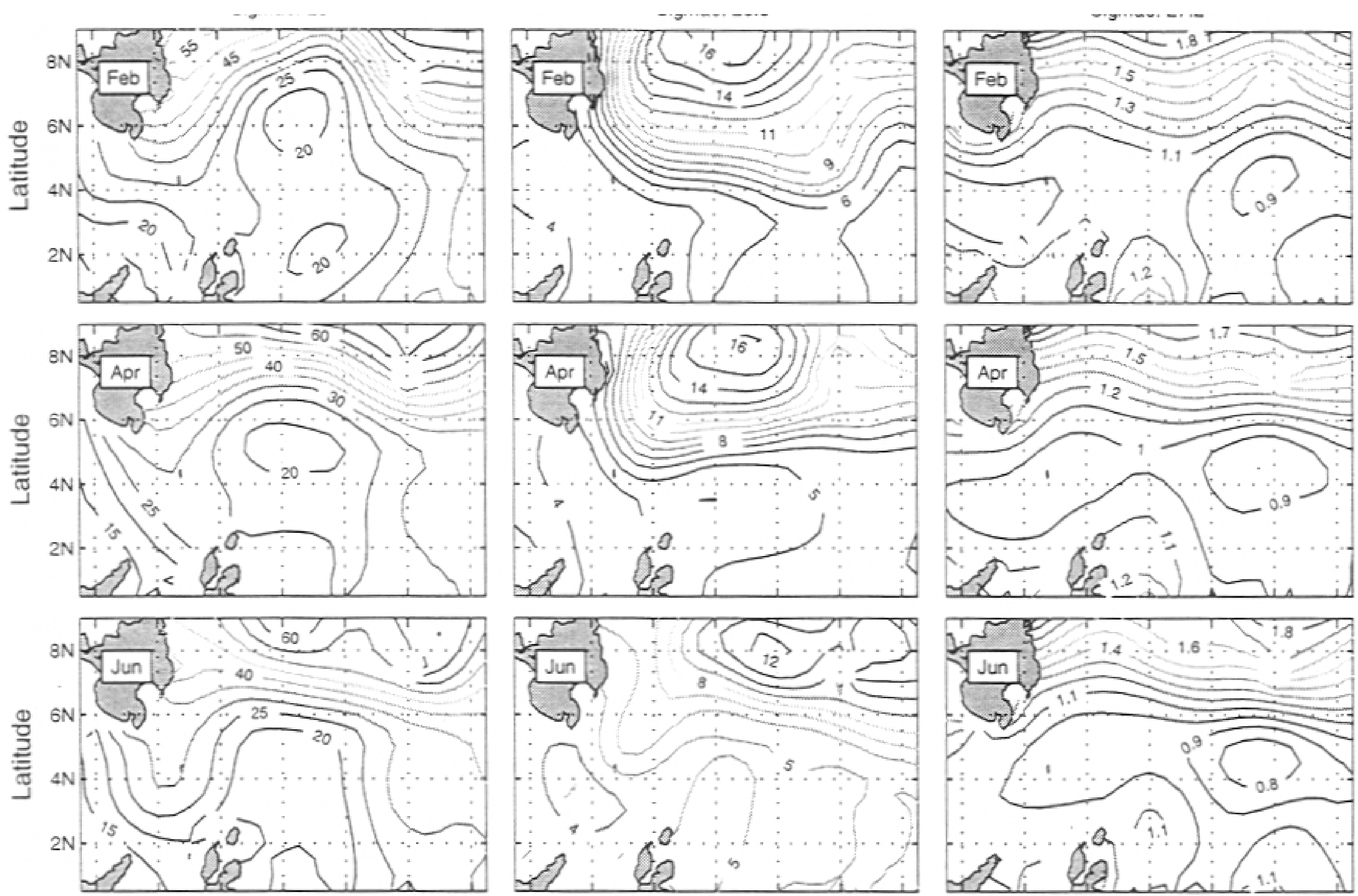


Salinity (GDEM)



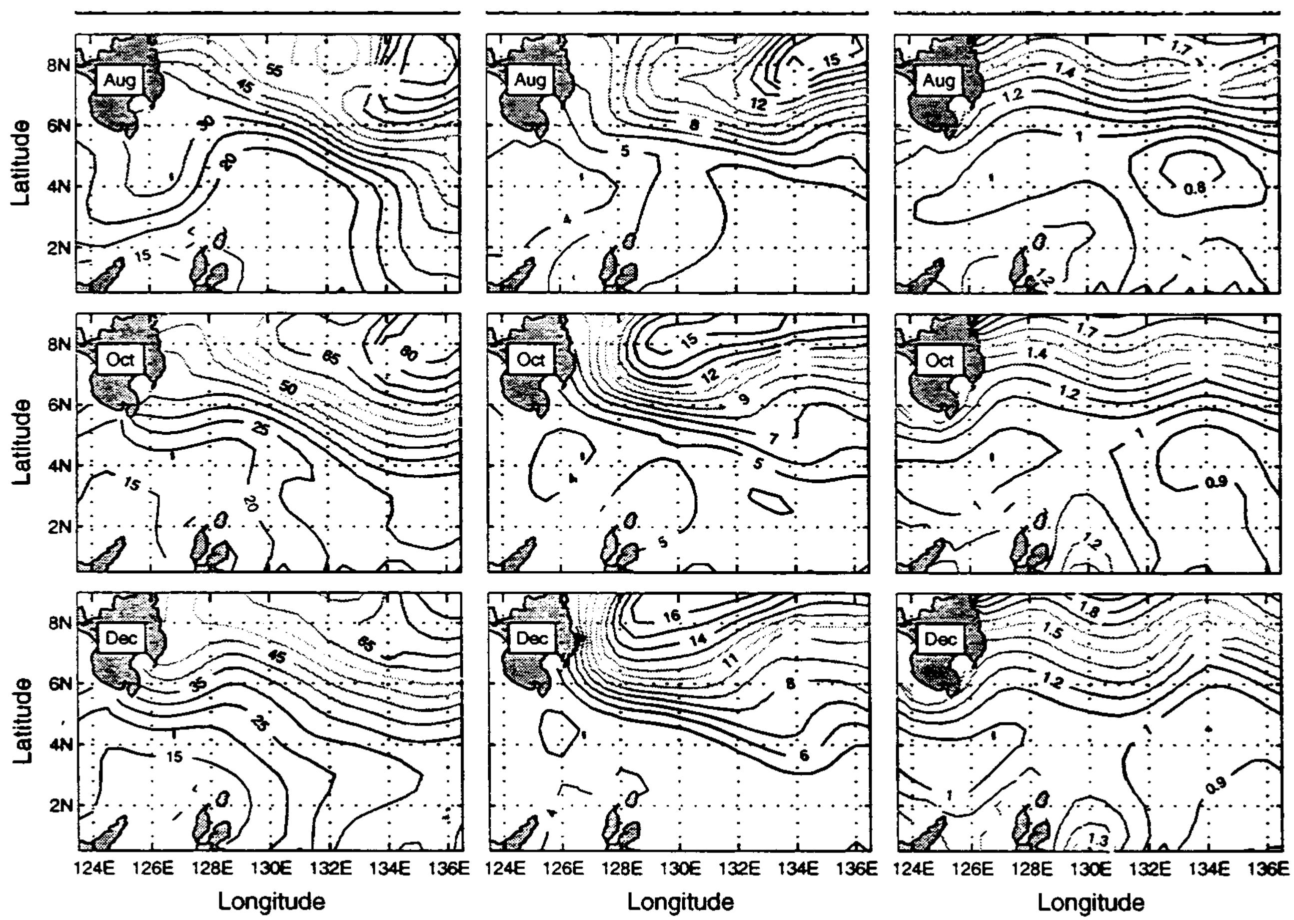
Potential Vorticity on Isopycnal Surfaces

- I



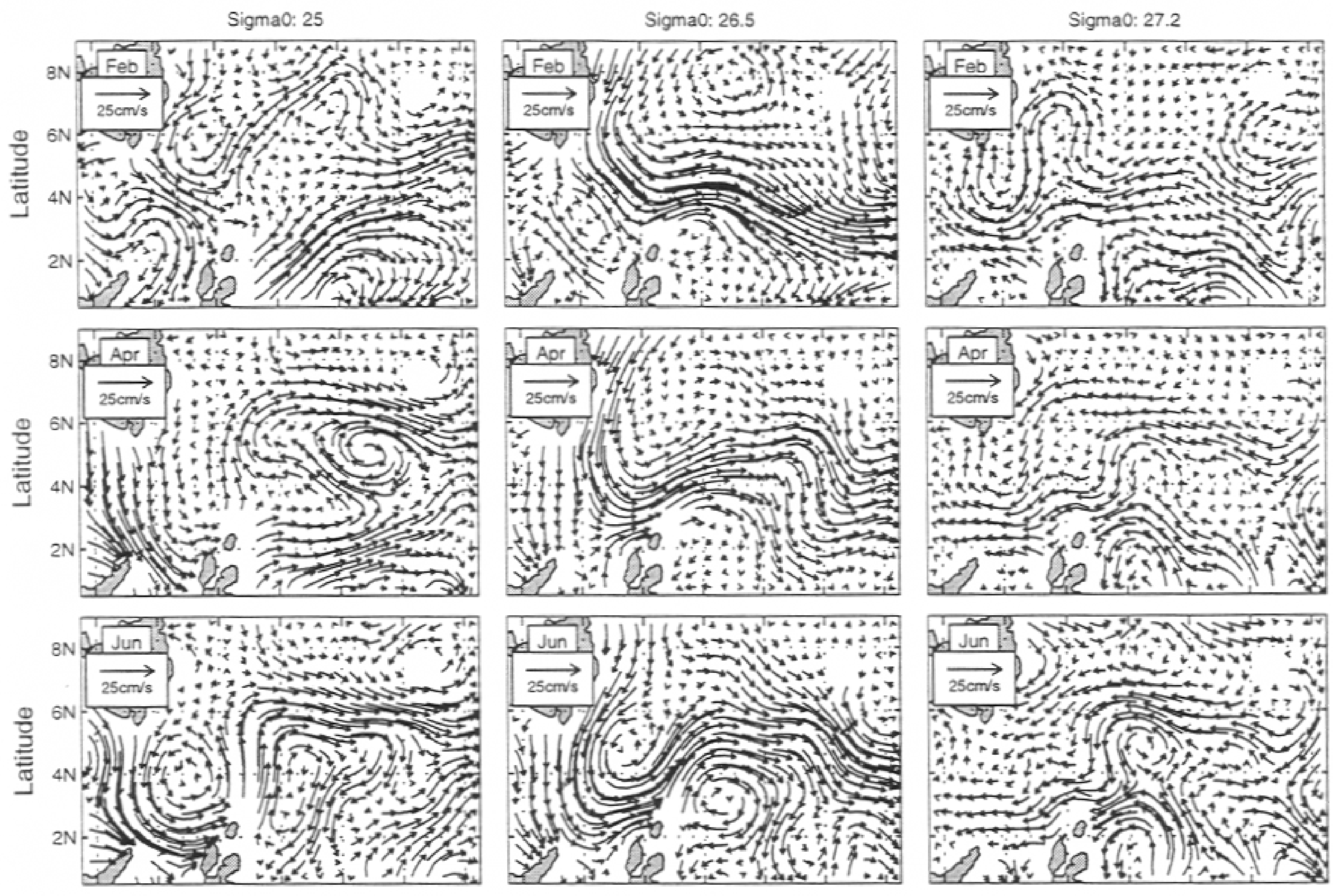
Potential Vorticity on Isopycnal Surfaces

- Isopycnal: $\sigma_\theta=25$ $\sigma_\theta=26.5$ $\sigma_\theta=27.2$



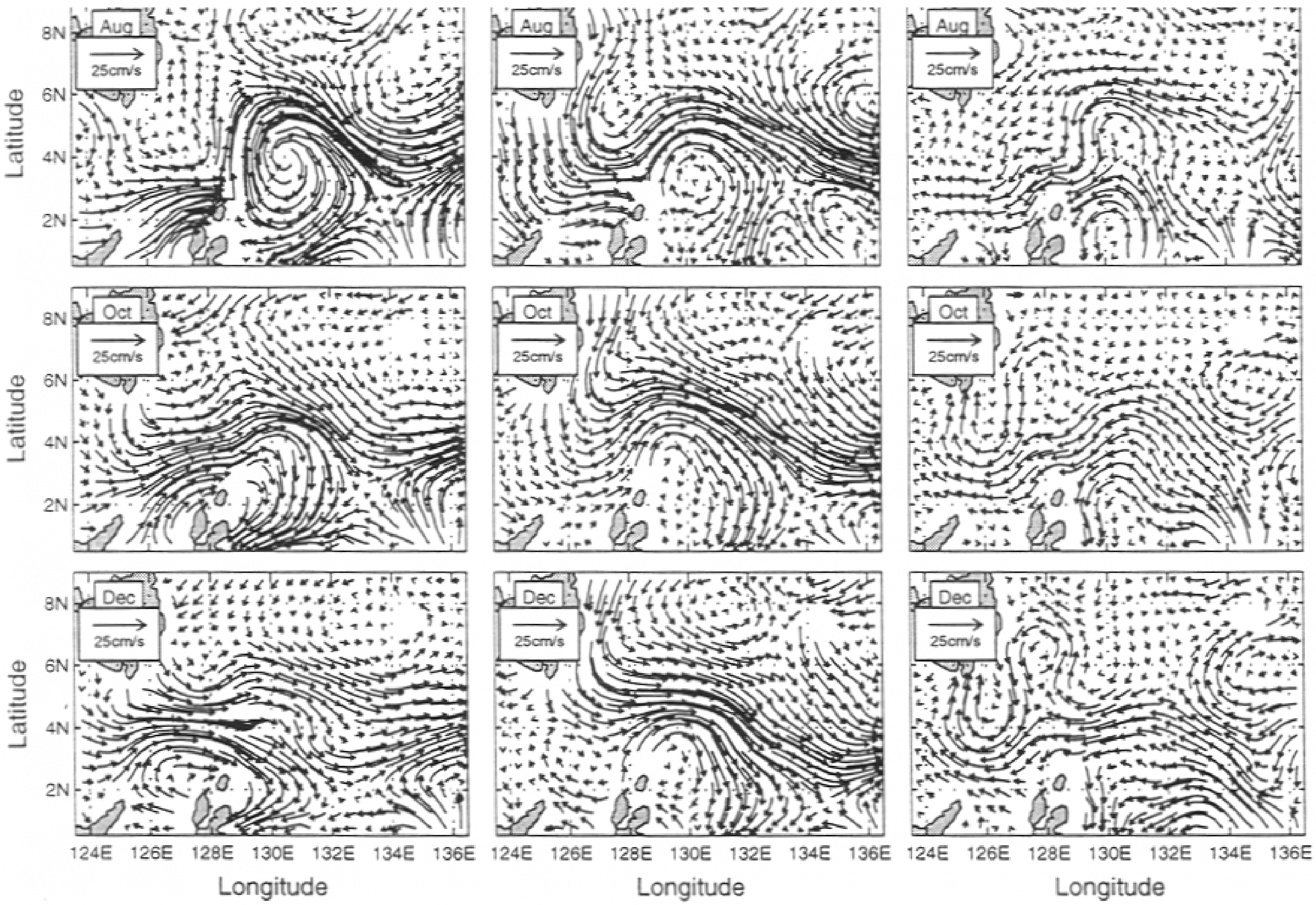
Circulation on Isopycnal Surface

- Isopycnal: $\sigma_\theta=25$ $\sigma_\theta=26.5$ $\sigma_\theta=27.2$

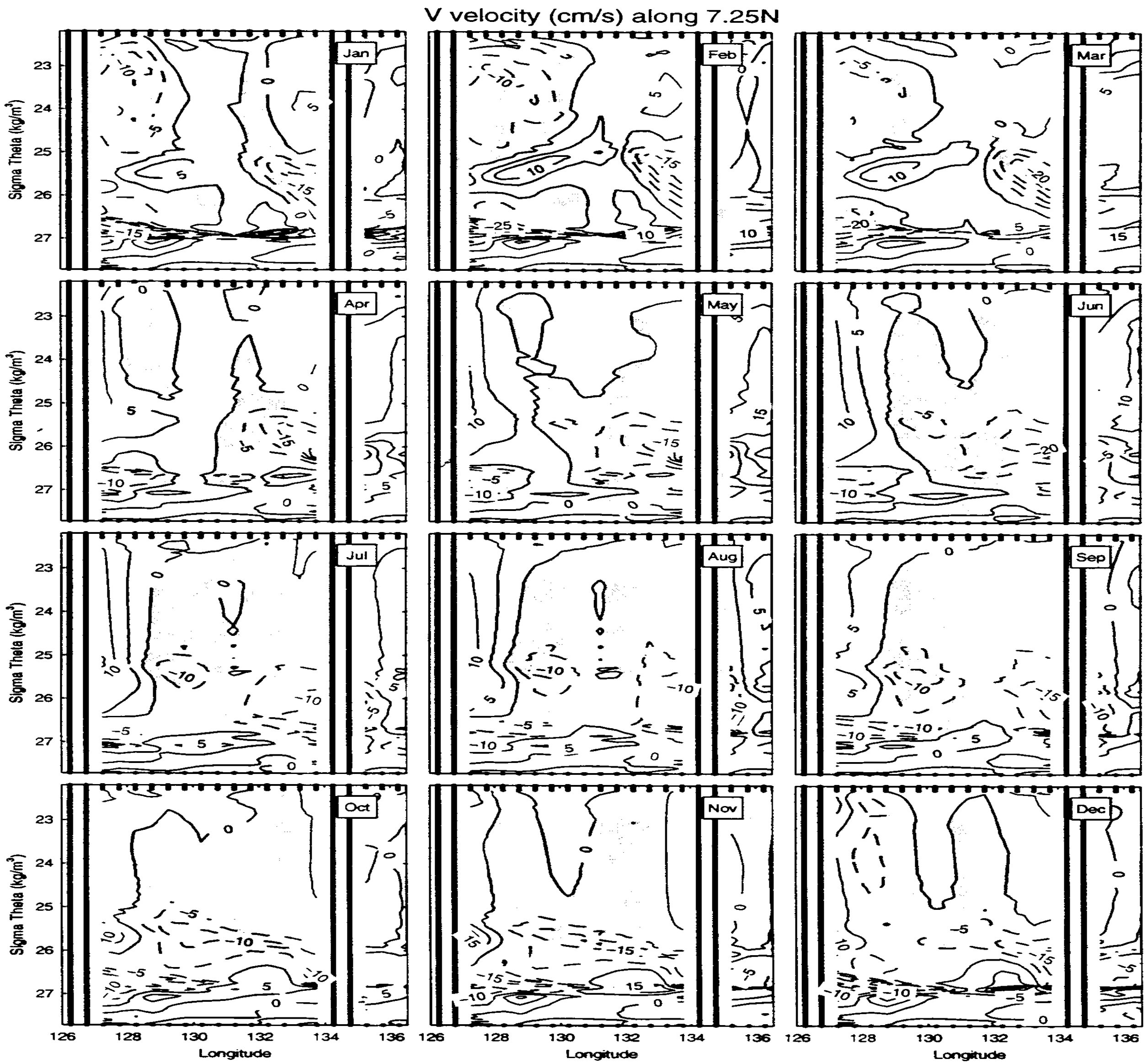


Circulation on Isopycnal Surface

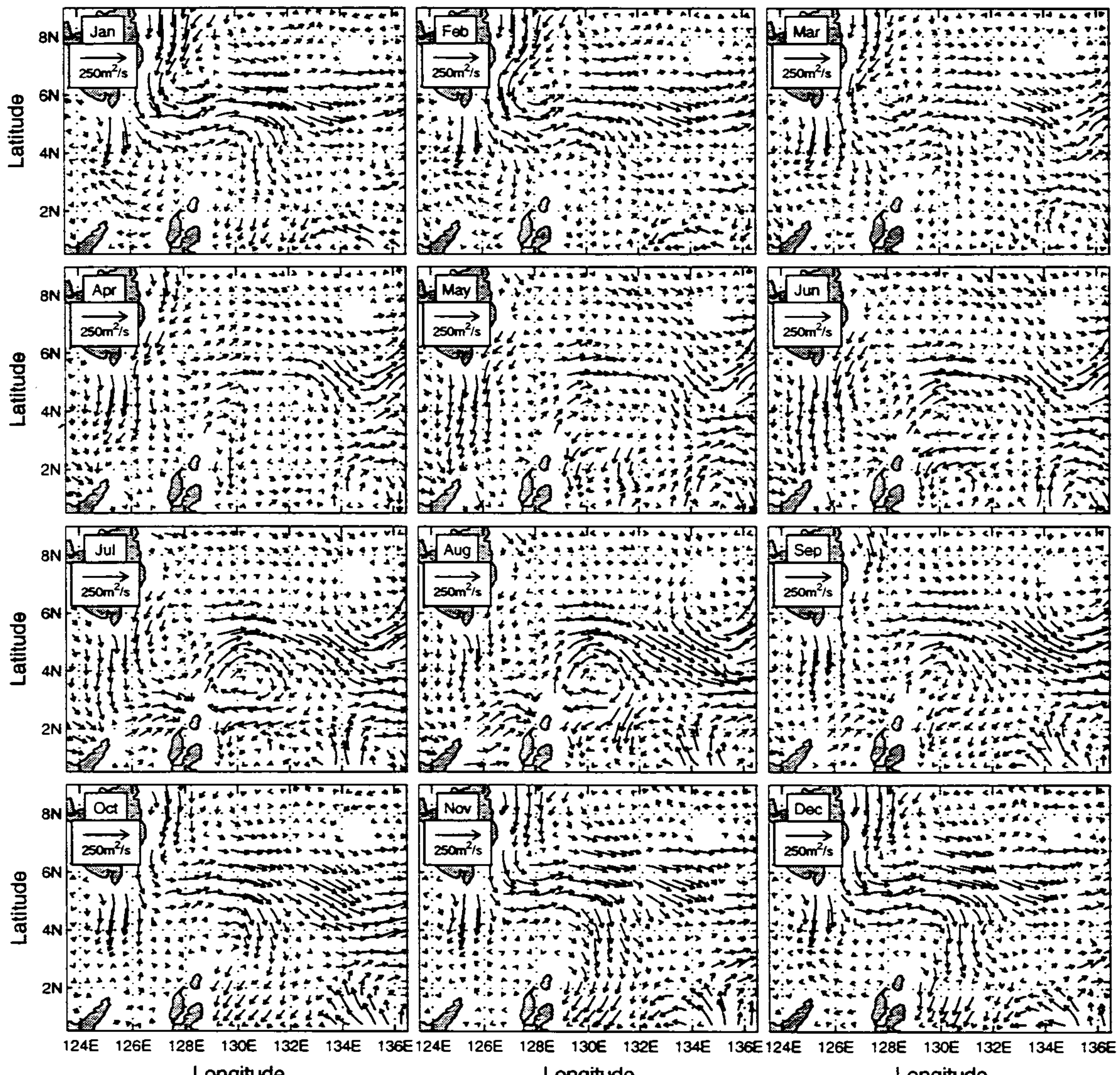
- Isopycnal: σ_θ=25 σ_θ=26.5 σ_θ=27.2



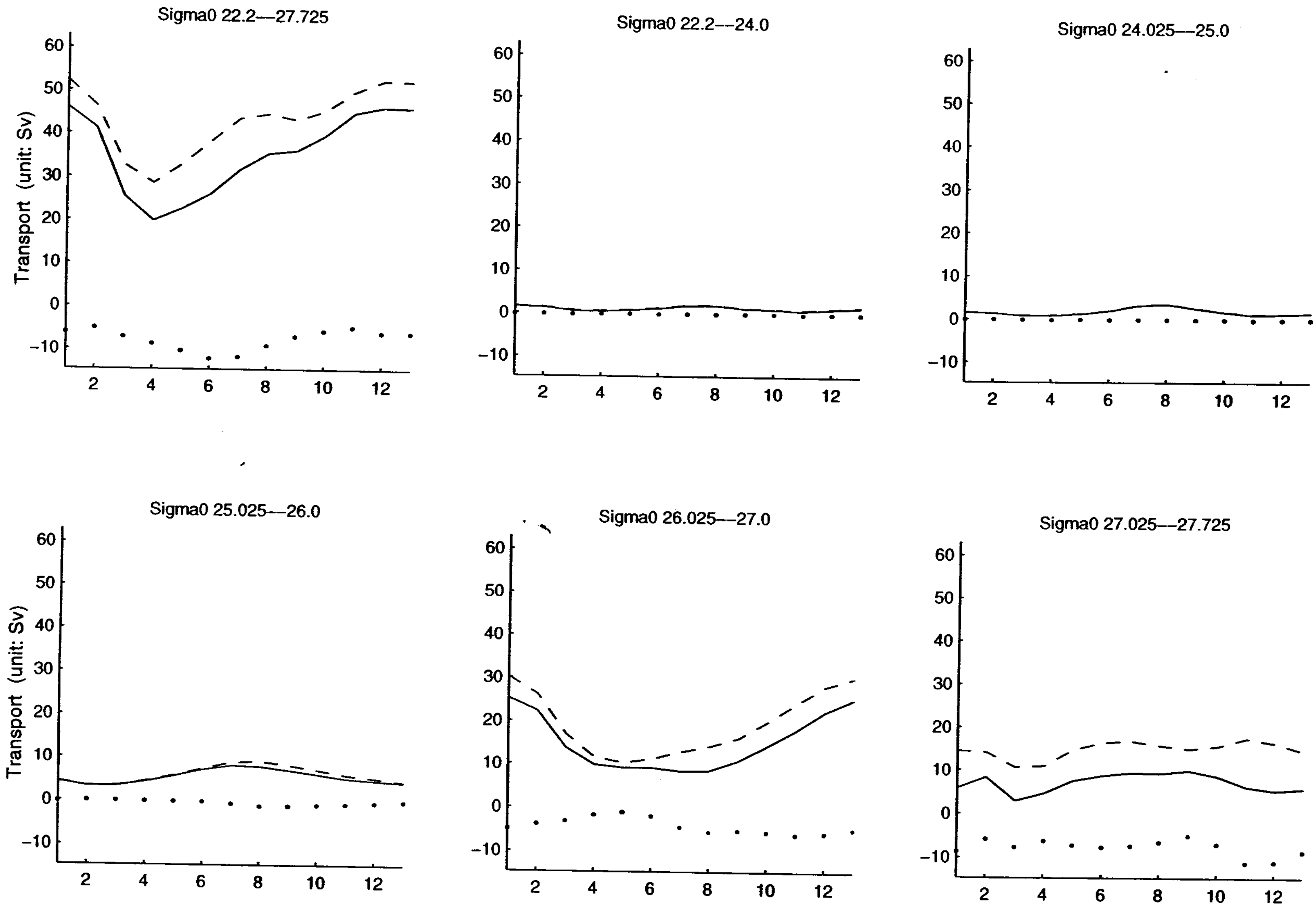
V-Velocity (cm/s) along 7.25^o N



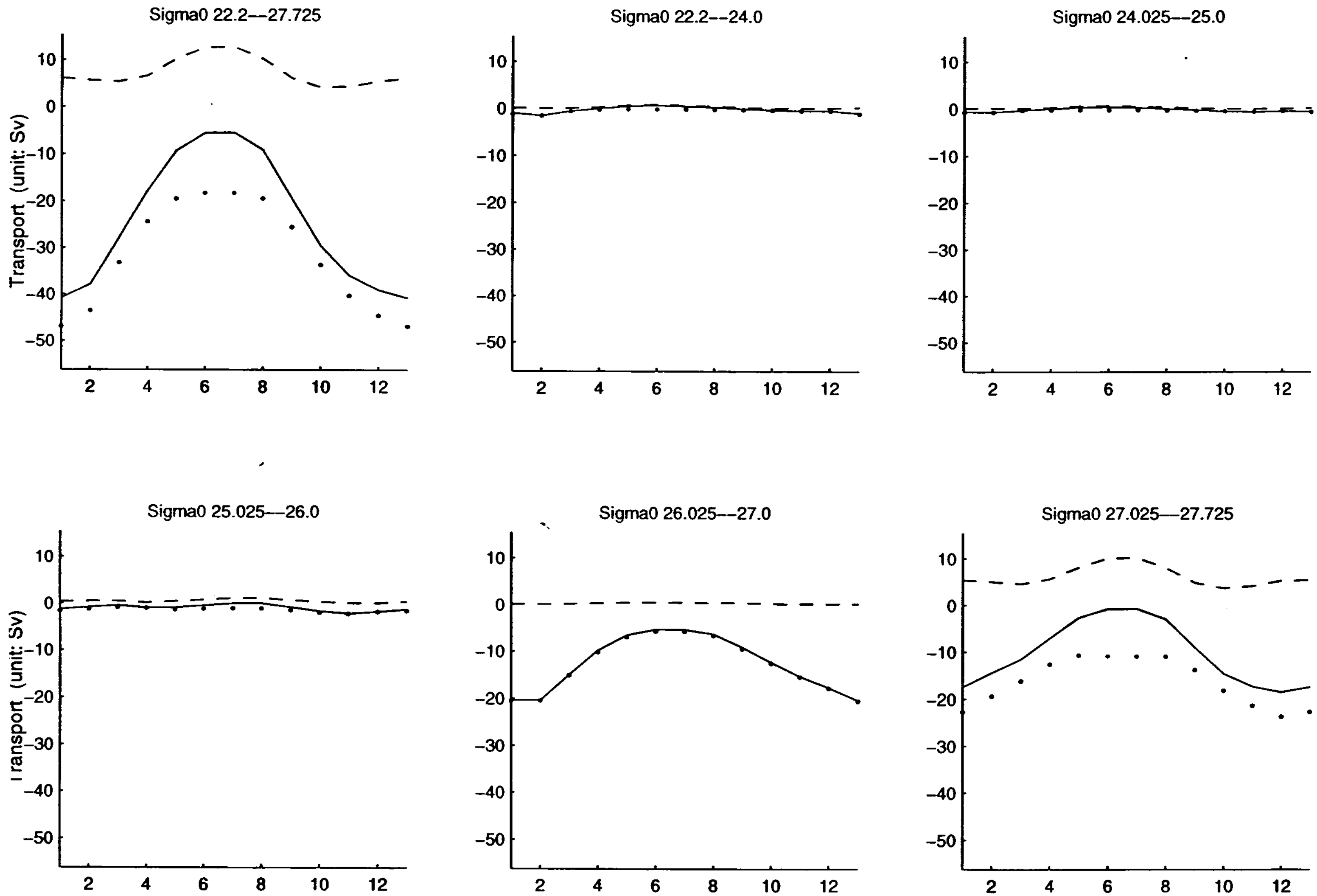
Vertically integrated velocity vectors (250 m^2 /s)



Volume Transport (Sv) of North Pacific Equatorial Counter Current (0.75 to 8.25° N) along 130.25° E



Volume Transport (Sv) of Midanao Current (126.75 to 130.75° E) along 8.25° N



- Seasonal Variability of Four Major Currents

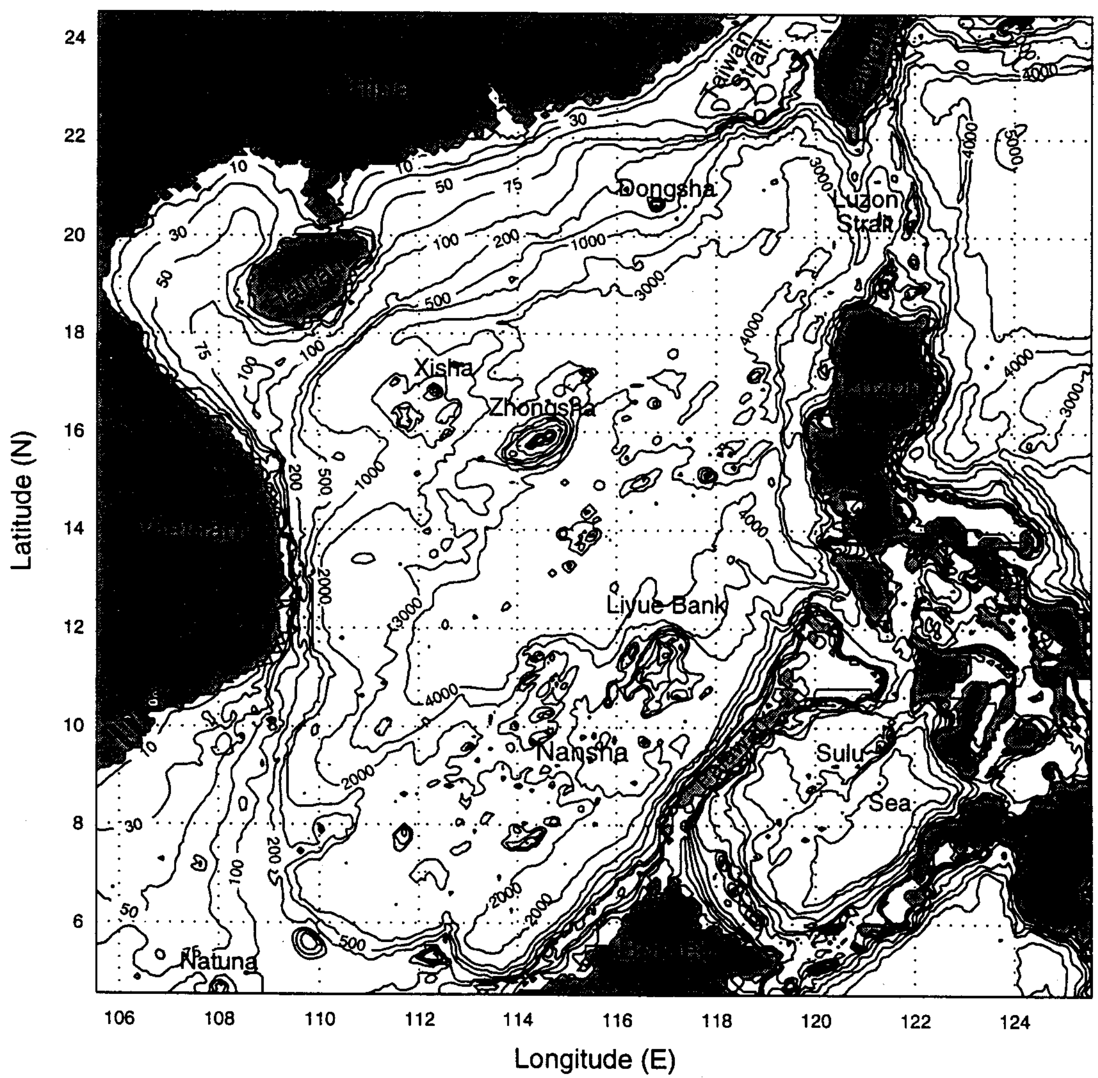
  Mindanao Current, Mindanao Counter Current, New Guinea Coastal Undercurrent, and North Equatorial Counter Current

  Two Eddies

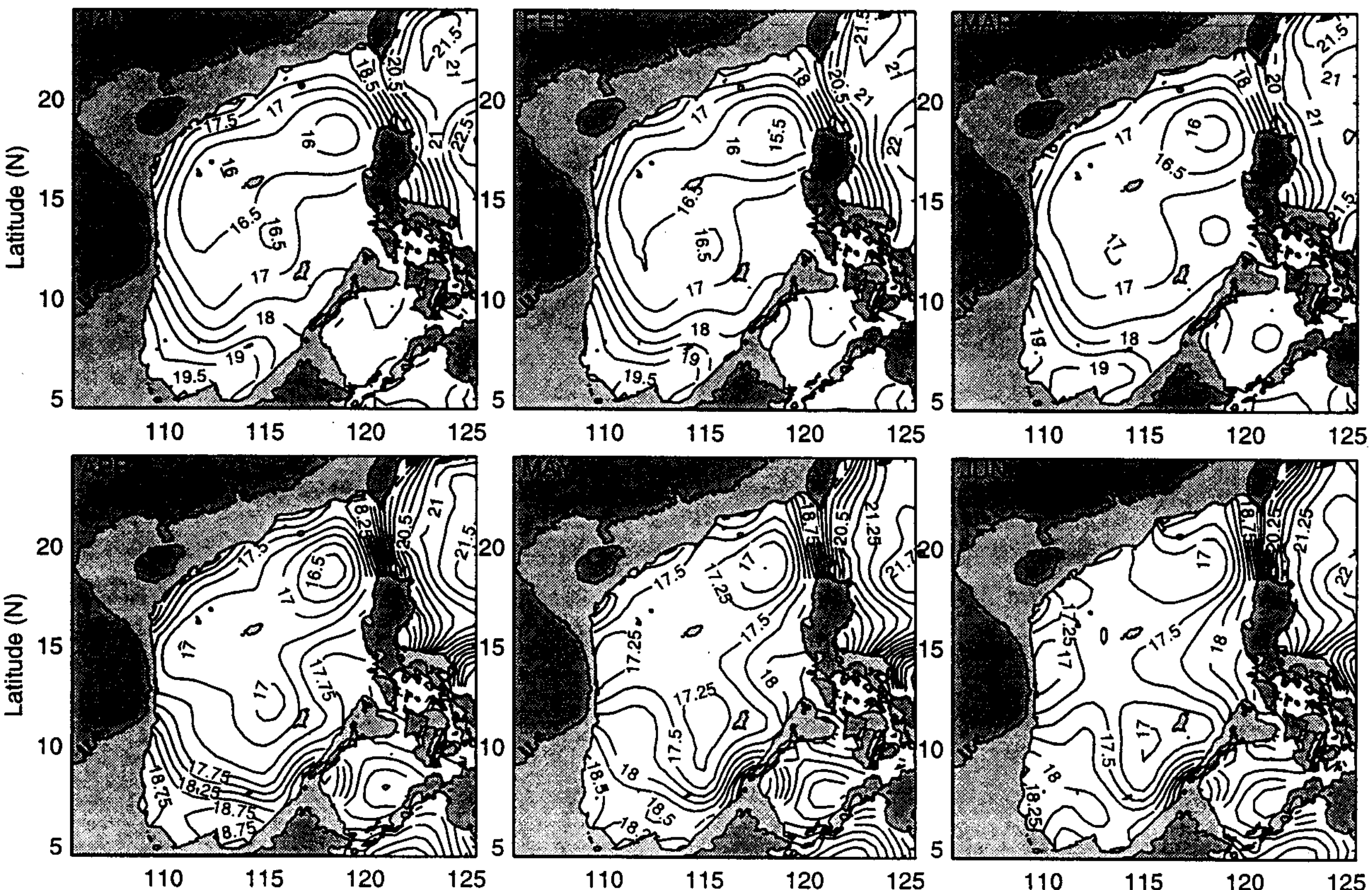
  Mindanao Eddy and Halmahera Eddy

Example-2
South China Sea Isopycnal Circulation Determined from GDEM

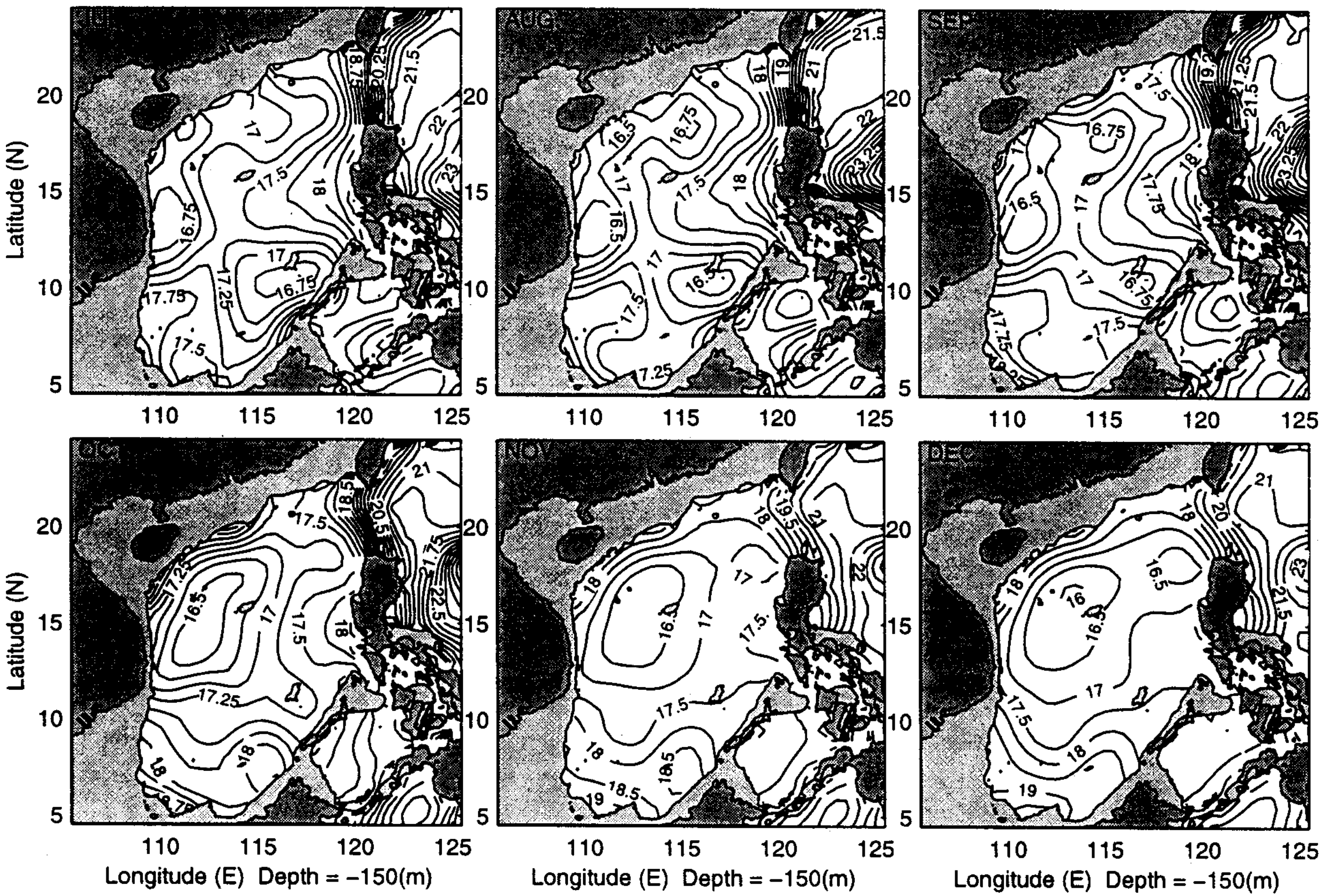
South China Sea



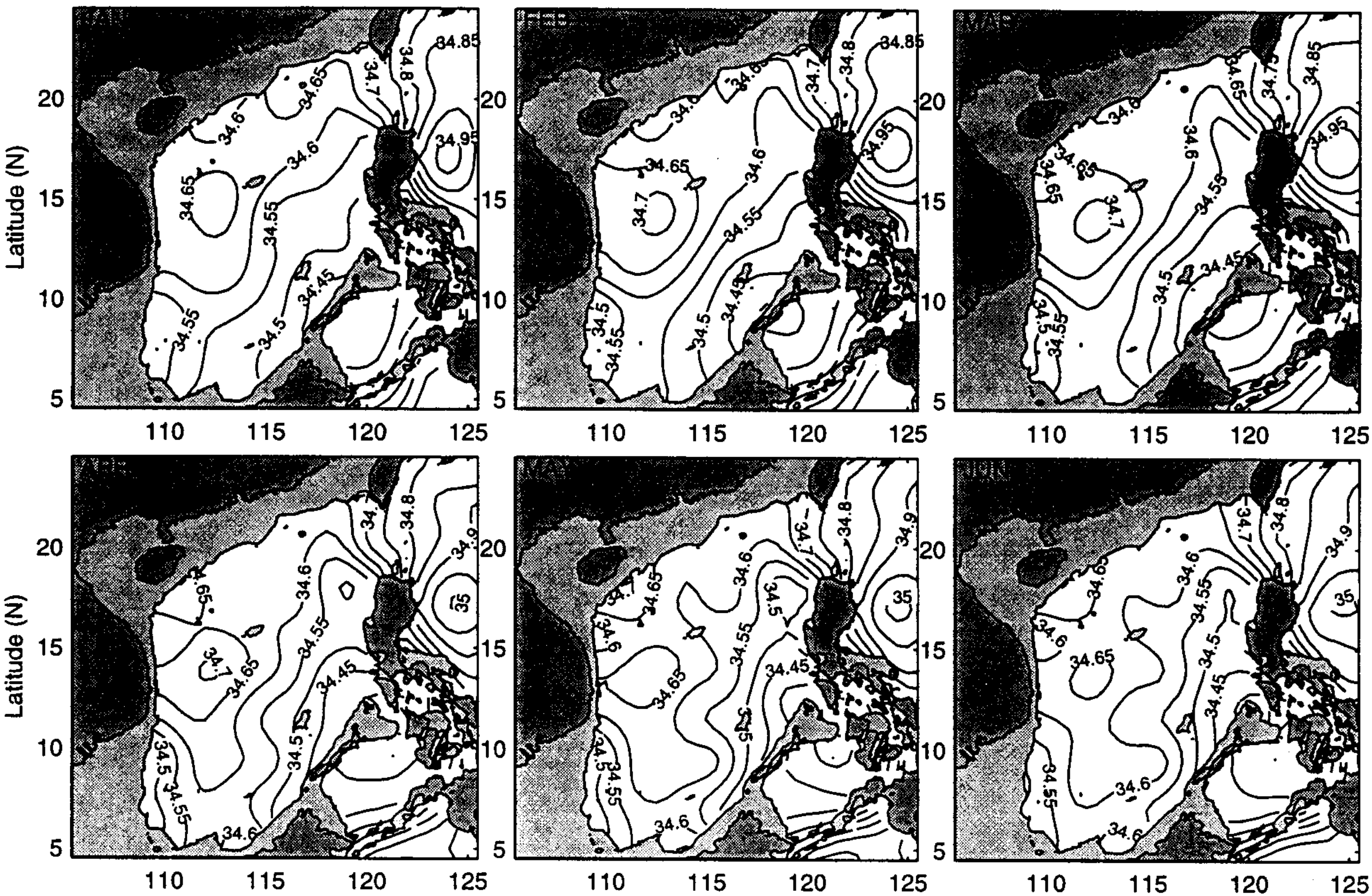
Monthly Mean T (150 m)



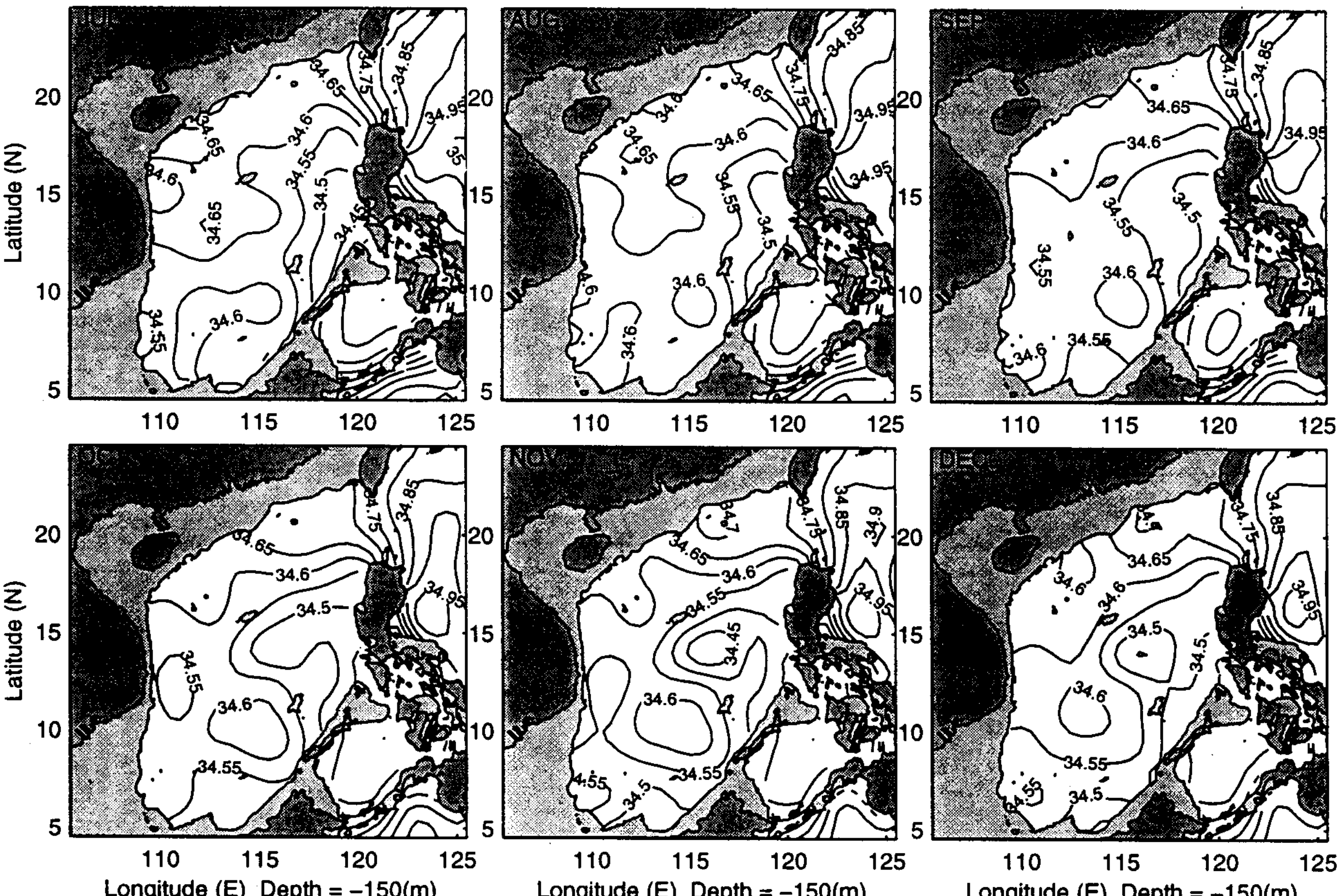
Monthly Mean T (150 m)



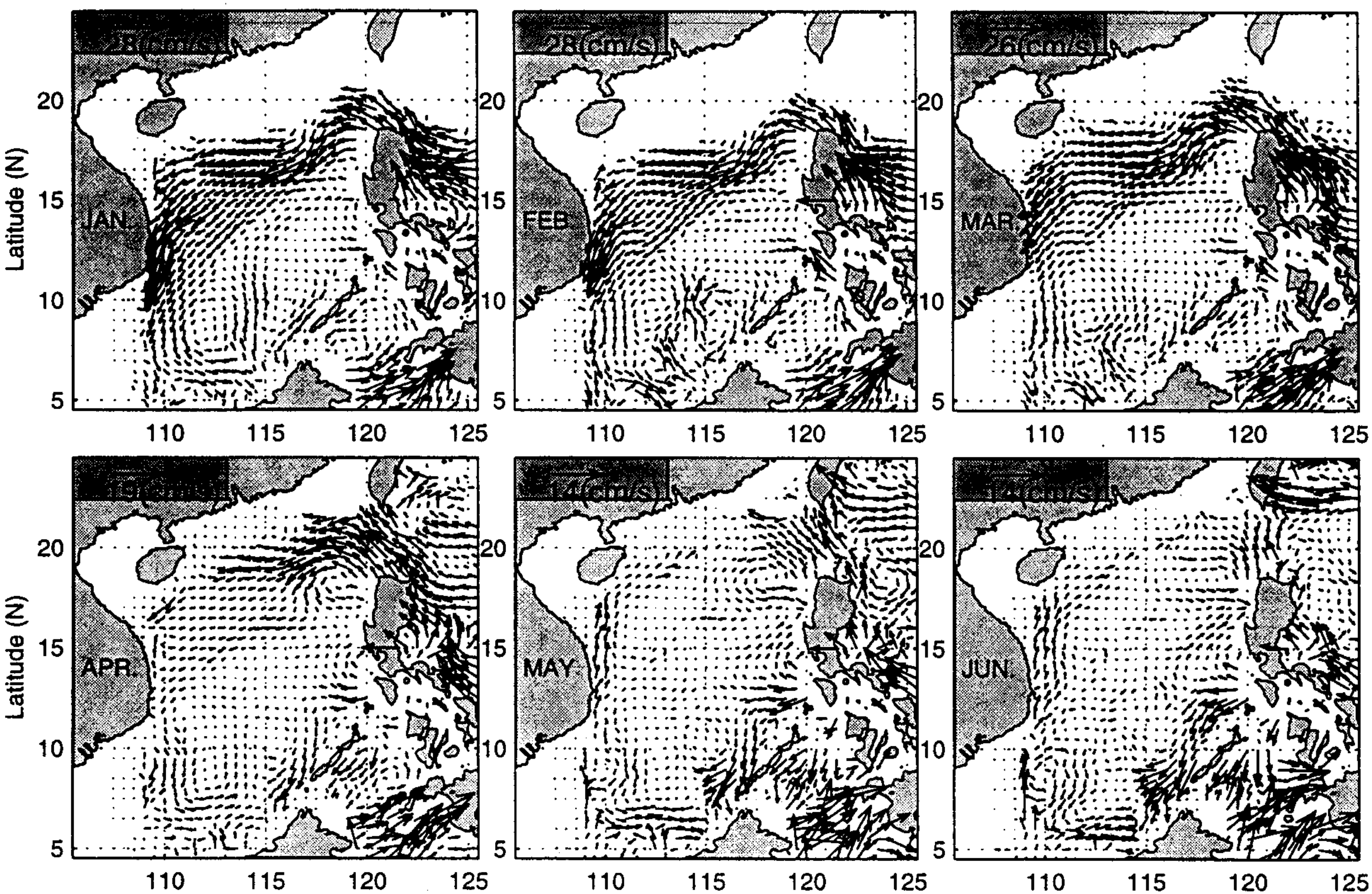
Monthly Mean S (150 m)



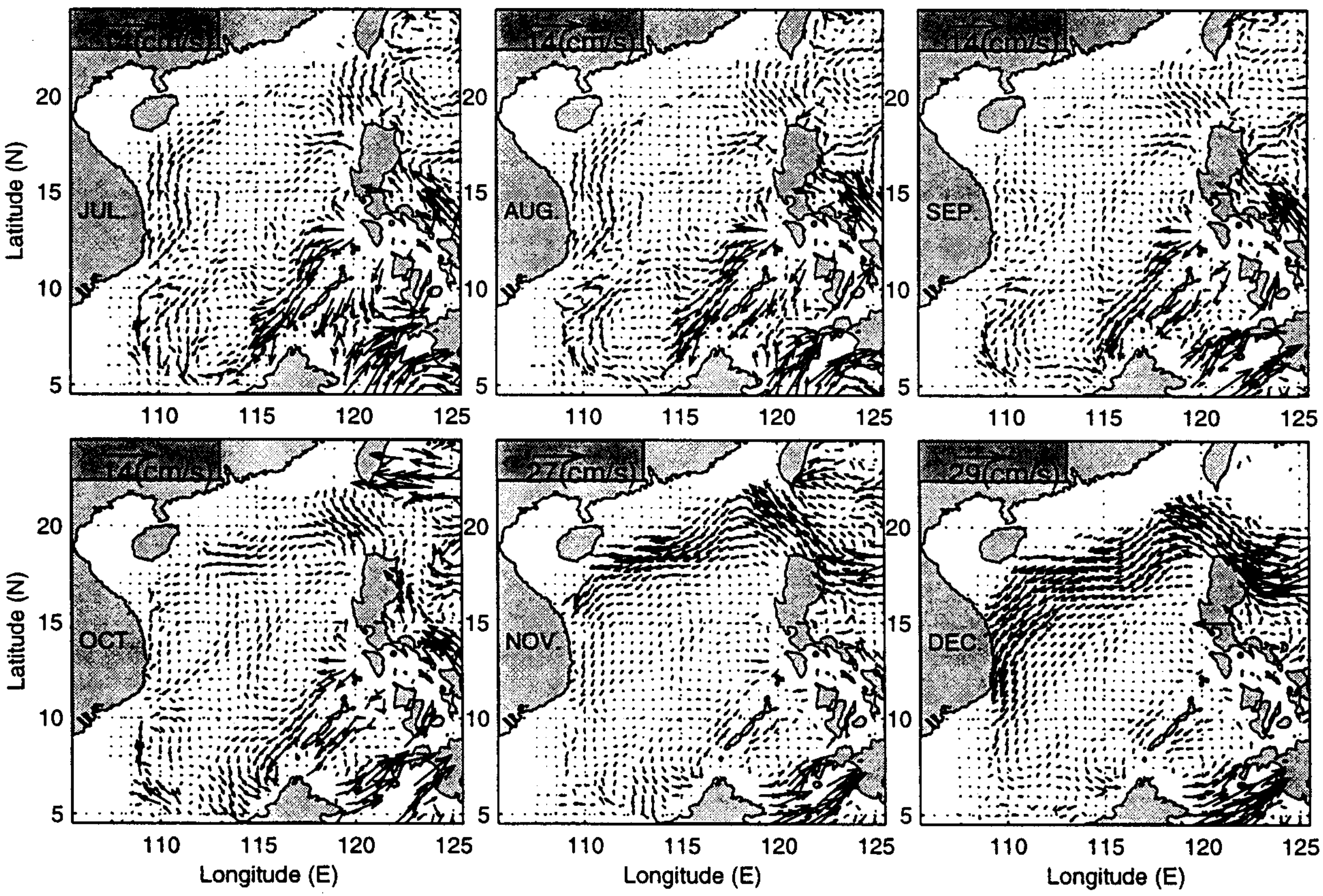
Monthly Mean S (150 m)



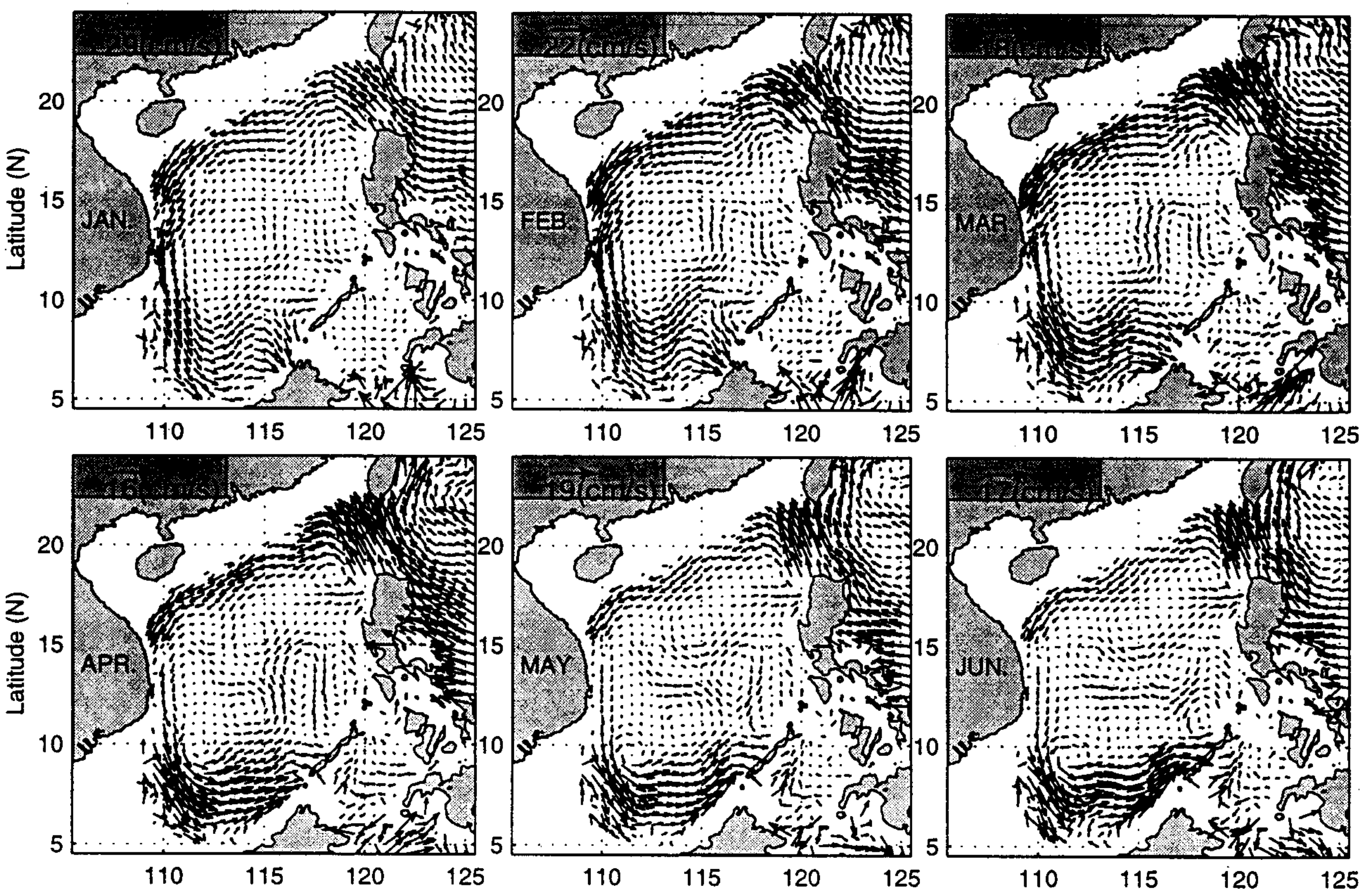
Monthly Mean Sub-Surface ($\sigma_\theta = 23.0$) Velocity Vector Field



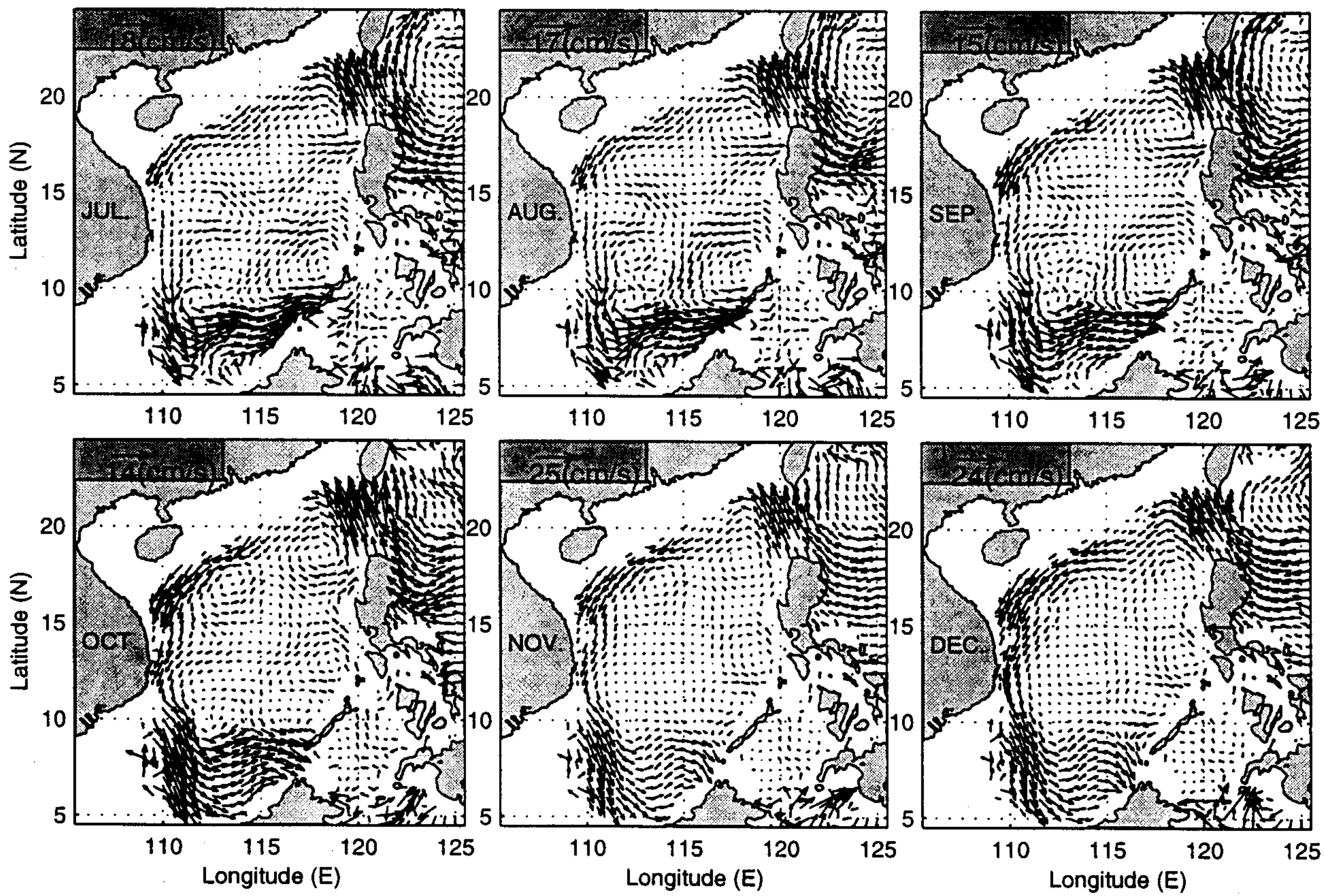
Monthly Mean Sub-Surface ($\sigma_\theta = 23.0$) Velocity Vector Field



Monthly Mean Intermediate Level ($\sigma_\theta = 26.2$) Velocity Vector Field

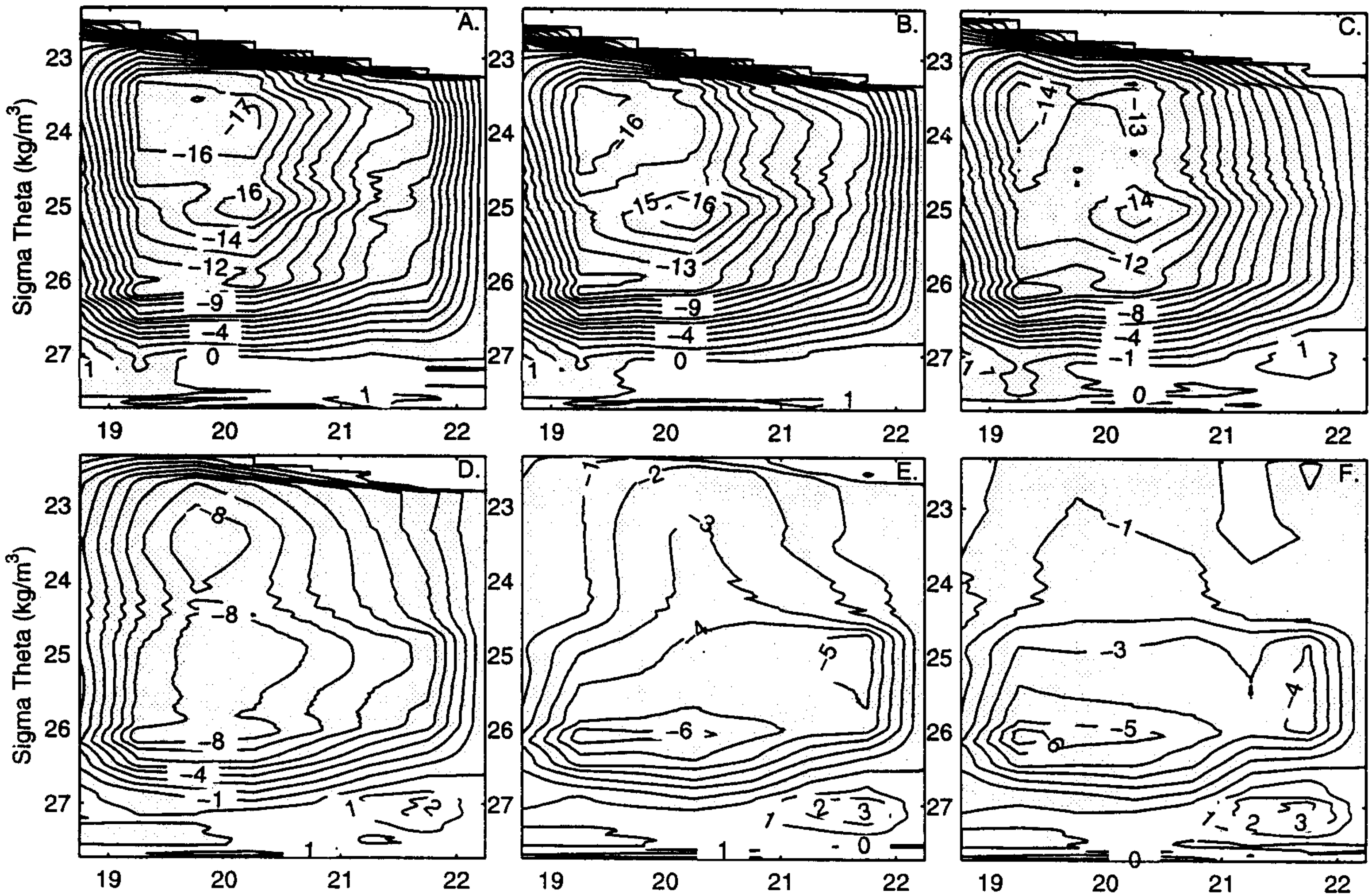


Monthly Mean Intermediate Level ($\sigma_\theta = 26.2$) Velocity Vector Field



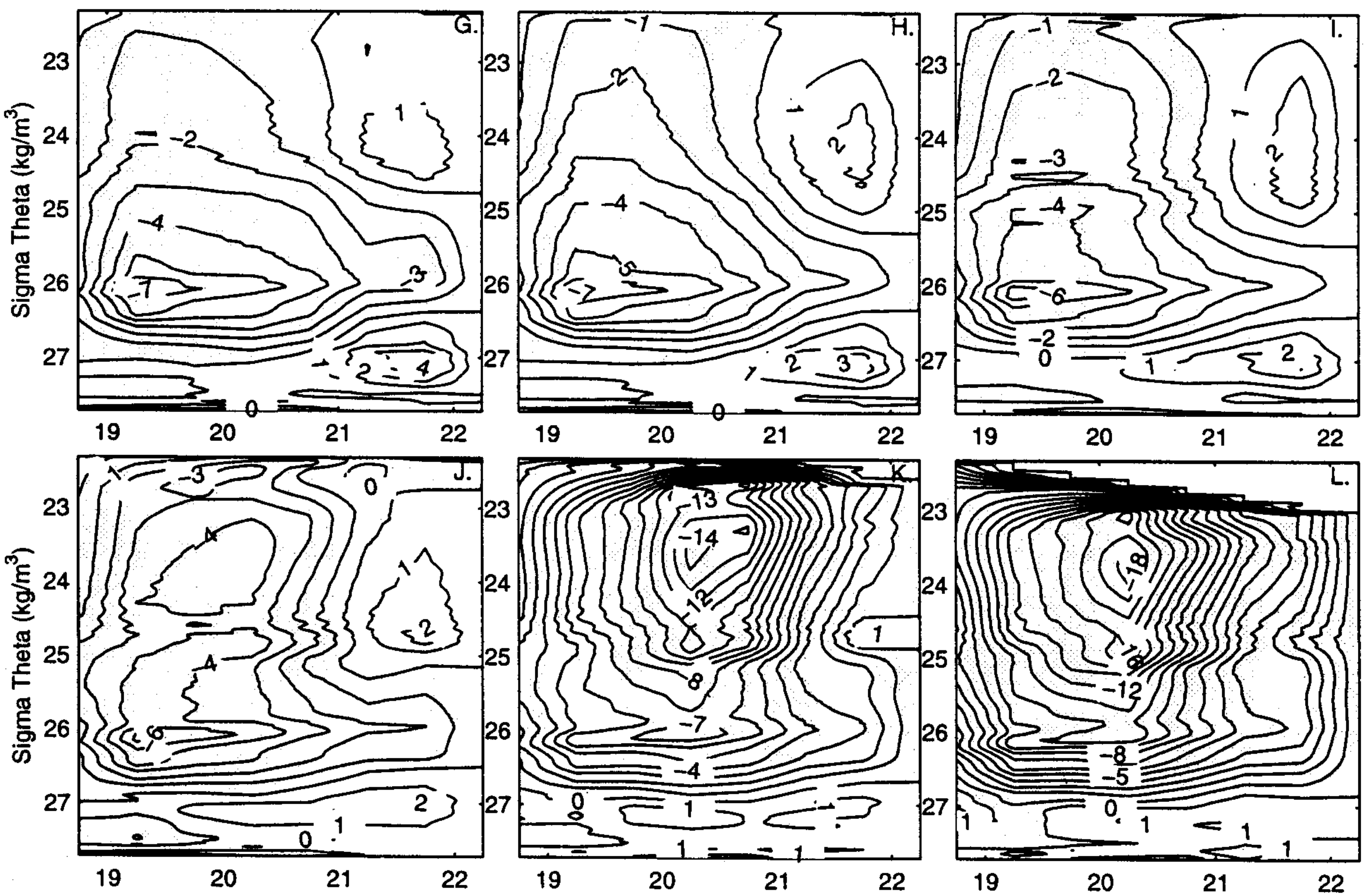
Monthly Mean Velocity Across Luzon Strait (Eastward Positive)

(A) Jan, (B) Feb, (C) Mar, (D) Apr, (E) May, (F) June



Monthly Mean Velocity Across Luzon Strait (Eastward Positive)

(G) Jul, (H) Aug, (I) Sep, (J) Oct, (K) Nov, (L) Dec



Conclusions (Part-3)

- The isopycnal surface circulation can be effectively determined using the P-Vector method.